Discrete Applied Mathematics 376 (2025) 270–280

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A note on extension properties and representations of matroids[☆]

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ARTICLE INFO

Article history:
Received 10 April 2024
Received in revised form 24 April 2025
Accepted 22 June 2025
Available online 30 June 2025

Keywords:
Matroid representation
Common information
Generalized Euclidean property
Secret sharing schemes

ABSTRACT

We discuss several extension properties of matroids and polymatroids and their application as necessary conditions for the existence of different matroid representations, namely linear, folded linear, algebraic, and entropic representations. Iterations of those extension properties are checked for matroids on eight and nine elements by means of computer-aided explorations, finding in that way several new examples of non-linearly representable matroids. A special emphasis is made on sparse paving matroids on nine points containing the tic-tac-toe configuration. We present a new, more clear description of that family and we analyze extension properties on those matroids and their duals.

1. Introduction

Given a linear representation of a matroid M, every flat determines a vector subspace. Consider a non-modular pair (F_1, F_2) of flats of M, that is, the rank of $F_1 \cap F_2$ is less than the dimension of the intersection of the corresponding subspaces. Then the rank of $F_1 \cap F_2$ is increased in some single-element extension of M that is still linearly representable. A necessary condition for a matroid to be linearly representable is derived from that fact, namely the *generalized Euclidean intersection property* discussed in [4]. In particular, it is not satisfied by the *Vámos matroid*.

A necessary condition for a matroid to be algebraic is given by the Ingleton–Main lemma [30]. It states that, given three non-coplanar lines in an algebraic matroid such that every two of them are coplanar, there is a single-element extension in which the three lines meet in one point and, moreover, the extension is algebraic. More general versions of that property were presented by Lindström [35] and Dress and Lovász [21]. The Ingleton–Main lemma provided the first example of a non-algebraic matroid [30], namely the Vámos matroid. Lindström [35] used a generalization of it to prove that the class of algebraic matroids has infinitely many excluded minors. Hochstättler [28] proved that the dual of the *tic-tac-toe matroid* is not algebraic by using the Ingleton–Main lemma. That could be a counterexample proving that the class of algebraic matroids is not closed by duality, but this is still an open problem.

Using the terminology introduced in [11], those necessary conditions for a matroid to be linear or algebraic are examples of *extension properties* of matroids. Each of them consists of a matroid extension with certain constraints. Every matroid in the class of interest admits extensions with the required properties that are in the same class.

☆ Michael Bamiloshin received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 713679 and from the Universitat Rovira i Virgili. Oriol Farràs is supported by grant 2021 SGR 00115 from the Government of Catalonia and by the project HERMES, funded by INCIBE and by the European Union NextGeneration EU/PRTR. Carles Padró is supported by the Spanish Government through grant PID2019-109379RB-I00. Additionally, the authors are supported by the project ACITHEC PID2021-124928NB-I00, funded by MCIN/AEI/10.13039/ 501100011033/FEDER, EU.

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https://doi.org/10.1016/j.dam.2025.06.028
0166-218X/

Extension properties of polymatroids have been used in information theory. Specifically, the Ahlswede–Körner lemma [1,2,32] and the copy lemma [17,32,46] determine extension properties of *almost entropic polymatroids*. These properties have been applied in the search for *non-Shannon linear information inequalities* [17,20,27,32,38], with the *Zhang–Yeung inequality* [46] being the first one to be found. *Non-Shannon linear rank inequalities* as, for example, the *Ingleton inequality* [29] are useful when dealing with discrete *linear* random variables. Almost all known such inequalities follow from the *common information property* [19], an extension property of linearly representable polymatroids.

Linear information and rank inequalities have been used as constraints in linear programs providing bounds on the information ratio of *secret sharing schemes* [8,9,36,42] and on the achievable rates in *network coding* [18,44,45]. Some of those bounds have been recently improved by using the aforementioned extension properties of polymatroids, from which information and rank inequalities are derived [6,22,26]. The idea is to substitute those inequalities in the linear programs by constraints that are derived from extension properties. In that way, it is not necessary to guess what inequalities fit best with a given problem. Among other results, those improved bounds made it possible to determine the optimal value of the information ratio of *linear* secret sharing schemes for all access structures on five players and all graph-based access structures on six players [22], partially concluding the projects initiated in [16,31].

Frobenius flocks, which have been introduced by Bollen [11], provide a remarkable new tool to find out the existence of algebraic representations of matroids. Exhaustive searches on small matroids have been carried out in [11] by using Frobenius flocks as well as Ingleton–Main and Dress–Lovász extension properties.

Building on the results by Mayhew and Royle [40] and their online database of matroids [43] and using, among other tools, common information and Ahslwede–Körner extension properties, the classification of matroids on small ground sets according to their representations was pursued in [6]. In addition to linear and algebraic, almost entropic and *folded linear* representations were considered. Those two classes of matroids play an important role in the theory of secret sharing schemes because of the Brickell–Davenport theorem [14]. Moreover, the interest of the former is increased by a recent result by Matúš [39], namely, algebraic matroids are almost entropic. Therefore, extension properties of almost entropic polymatroids apply to algebraic matroids too.

In contrast to previous works [4,21,28,30,35], more recent applications of extension properties [6,11,22,26] have been obtained by computer-aided explorations in which several iterations of the chosen extension are searched. On the negative side, due to their computational complexity, those explorations are only feasible for matroids and polymatroids on small ground sets.

The classification of matroids on eight elements is almost concluded by the results in [6,11,40]. Only for three of them it is not known whether they are algebraic, almost entropic, or neither. There are exactly 39 matroids on eight elements that do not satisfy the Ingleton inequality [40]. They are *sparse paving* matroids and each of them contains five *circuit-hyperplanes* in the same configuration as the ones of the Vámos matroid. Because of that, those matroids do not satisfy the Ahlswede–Körner property, and hence they are neither algebraic nor almost entropic. Moreover, all of them are relaxations of the *maximal* sparse paving matroid $AG(3,2)$.

One of the main open problems about the classification of matroids on nine elements is to determine whether the tic-tac-toe matroid T^3 is algebraic or not. Its dual matroid does not satisfy the Ingleton–Main extension property [28], and hence it is not algebraic. Therefore, T^3 is a candidate for a counterexample proving that the class of algebraic matroids is not closed by duality.

1.1. Our results

In this work, we pursue the application of extension properties to the classification of small matroids according to their representations. We provide a unified view of extension properties previously studied in the area of Matroid Theory (i.e., Euclidean, generalized Euclidean, Levi's intersection, and Ingleton–Main) with others studied in the area of Information Theory (common information and Ahlswede–Körner) that help us combine theoretical results and computer-aided tools for the classification of representable matroids. We discover a new family of matroids connected to the tic-tac-toe matroid that are not representable. We mainly focus on sparse paving matroids on eight and nine elements.

In the first stage of this work, we carried out computer-aided explorations on the databases of matroids by Royle and Mayhew [43] and Bollen [12]. Using linear programs in a blanket approach to check the existence of iterated extension properties for matroids on nine points appeared to be too computationally costly. Instead, we checked the feasibility of iterated generalized Euclidean extensions by exploring the existence of the associated modular cuts [41, Section 7.2]. Once impossible iterated generalized Euclidean extensions are found, one can check by linear programming other extension properties on those particular situations, like common information and Ahlswede–Körner. That strategy provided several new examples of non-linear matroids, which are listed in the extended version of this paper [7]. The computer programs are available in [5].

The outcome of those explorations lead us to analyze in more detail the sparse paving matroids that are relaxations of P_8 and the ones that present the tic-tac-toe configuration.

There are only five matroids on eight elements that satisfy the Ingleton inequality but are not linear. Two of them are folded linear [6] and algebraic [13,34]. The other three, which are relaxations of the maximal sparse paving matroid P_8, are not folded linear [6] and it is not known whether they are algebraic, almost entropic, or neither. We checked by computer-aided explorations that they do not satisfy the common information property. Therefore, the same applies to the generalized Euclidean property. For two of those matroids, we present a human readable proof for that fact. In particular, the Vámos configuration appears after some generalized Euclidean extensions, which may be a hint indicating that those matroids are not algebraic.

In addition to the configurations of the Vámos matroid and the aforementioned relaxations of P_8, the configurations of the circuit-hyperplanes of the tic-tac-toe matroid T^3 and its dual are also of interest in regard to extension properties. The matroid T^3 does not satisfy the generalized Euclidean property [3], but it satisfies the Ingleton–Main and Dress–Lovász properties [11]. In contrast, its dual matroid does not satisfy the Ingleton–Main property [28]. Similarly to the Vámos configuration, we show that this configuration of circuit-hyperplanes of the tic-tac-toe matroid is also a configuration that prevents linear representability of a sparse paving matroid. We present a new, clear and complete description of the sparse paving matroids on nine elements that contain this configuration. In particular, we identify two maximal sparse paving matroids whose relaxations contain all of them. Finally, we prove that the dual tic-tac-toe matroids do not satisfy the Ahslwede–Körner property, and hence they are not almost entropic.

2. Representations of matroids and polymatroids

Basic facts on matroids and polymatroids and some different ways in which they can be represented are discussed in this section.

We begin by introducing some notation. The number of elements of a finite set X is denoted by $|X|$ and $\mathcal{P}(X)$ denotes its power set. Most of the times we use a compact notation for set unions and we avoid the curly brackets for singletons. That is, we write XY for $X \cup Y$ and Xy for $X \cup \{y\}$. In addition, we write $X \smallsetminus Y$ for the set difference and $X \smallsetminus x$ for $X \smallsetminus \{x\}$. For a *set function* $f : \mathcal{P}(E) \to \mathbf{R}$ on a finite set E and sets $X, Y, Z \subseteq E$, we denote

$$f(X;Y|Z) = f(XZ) + f(YZ) - f(XYZ) - f(Z)$$

and, in particular, $f(X;Y) = f(X;Y|\emptyset) = f(X) + f(Y) - f(XY)$ and $f(X|Z) = f(X;X|Z) = f(XZ) - f(Z)$. A set function f on E is *monotone* if $f(X) \le f(Y)$ whenever $X \subseteq Y \subseteq E$ and it is *submodular* if $f(X) + f(Y) - (X \cap Y) - f(X \cup Y) \ge 0$ for every $X, Y \subseteq E$.

Definition 2.1. A *polymatroid* is a pair (E, f) formed by a finite set E (the *ground set*) and a monotone and submodular set function f on E with $f(\emptyset) = 0$ (the *rank function*). In the particular case that f is integer-valued and $f(x) \le 1$ for every $x \in E$, the polymatroid (E, f) is a *matroid*.

A polymatroid (E, f) is *linearly representable over a field* K or K*-linearly representable*, or simply K*-linear*, if there exists a collection $(V_x)_{x\in E}$ of vector subspaces of a K-vector space V such that

$$f(X) = \dim \sum_{x\in X} V_x$$

for every $X \subseteq E$. That collection of subspaces is a K*-linear representation* of the polymatroid. If (E, f) is a matroid, then $\dim V_x \le 1$ for every $x \in E$, and we can replace each subspace V_x by a vector v_x that spans the subspace. That is, linear matroids are represented by collections of vectors. For a positive integer ℓ, a matroid (E, f) is ℓ-folded K*-linear* if the polymatroid $(E, \ell f)$ is K-linear. Every K-linear representation $(V_x)_{x\in E}$ of the polymatroid $(E, \ell f)$ is an ℓ-folded K*-linear representation* of the matroid (E, f).

Consider a field extension L/K and a collection $(e_x)_{x\in E}$ of elements in L. For every $X \subseteq E$, take $K(X) = K((e_x)_{x\in X})$ and let $f(X)$ be the transcendence degree of the field extension $K(X)/K$. Then (E, f) is a matroid. In that situation, (E, f) is *algebraically representable over* K or K*-algebraically representable*, or simply K*-algebraic*, and $(e_x)_{x\in E}$ is a K*-algebraic representation* of (E, f). Every K-linear matroid is K-algebraic.

The joint Shannon entropies of a collection of random variables determine the rank function of an *entropic* polymatroid [23,24]. Limits of entropic polymatroids are called *almost entropic*. It is well known that linearly representable polymatroids are almost entropic, see [19] for a concise proof. As a consequence, folded linear matroids are almost entropic. Matúš [39] proved that algebraic matroids are almost entropic too. Nevertheless, folded linear matroids are not necessarily algebraic [10]. For a graphical summary of the connections between the classes of matroids determined by their representations, see [6, Figure 1].

For a polymatroid (E, f) and disjoint sets $Z_1, Z_2 \subseteq E$, the *deletion* of Z_1 and the *contraction* of Z_2 results in the polymatroid on $E \smallsetminus Z_1Z_2$ with rank function $(f\backslash Z_1|Z_2)(X) = f(X|Z_2)$. Such polymatroids are the *minors* of (E, f). In particular, we denote $(f\backslash Z) = (f\backslash Z_1|\emptyset)$ and $(f|Z) = (f\backslash\emptyset|Z)$. Observe that minors of matroids are matroids. The *dual* of a matroid $M = (E, f)$ is the matroid $M^* = (E, f^*)$ with

$$f^*(X) = |X| - f(E) + f(E \smallsetminus X)$$

for every $X \subseteq E$. Each of the classes of matroids or polymatroids that are determined by the described representations is minor-closed. The class of linearly representable polymatroids and the class of folded linear matroids are duality-closed. It is unknown whether this applies to algebraic matroids or not, while the class of almost entropic polymatroids is not duality-closed [15,33].

We introduce next some additional terminology and basic facts about matroids. The *independent sets* of a matroid $M = (E, f)$ are those with $f(X) = |X|$. Every subset of an independent set is independent. The *bases* of M are the maximal independent sets, while the minimal dependent sets are called *circuits*. All bases have the same number of elements, which equals $f(E)$, the *rank of the matroid*. The *closure* of $X \subseteq E$ is formed by all elements $x \in E$ such that $f(Xx) = f(X)$. A *flat* is a set that is equal to its closure. Flats of rank 2, 3, or $f(E) - 1$ are called *lines*, *planes*, or *hyperplanes*, respectively.

A matroid of rank k is *paving* if the rank of every circuit is either k or $k-1$. It is *sparse paving* if, in addition, all circuits of rank $k-1$ are flats, which in that situation are called *circuit-hyperplanes*. Therefore, every sparse paving matroid M is determined by the family $\mathcal{C}_o(M)$ of its circuit-hyperplanes. Observe that every set in $\mathcal{C}_o(M)$ has exactly k elements and the intersection of any two different sets in $\mathcal{C}_o(M)$ has at most $k-2$ elements. Moreover, every family of sets with those properties determines a sparse paving matroid of rank k. If M and M' are sparse paving matroids of the same rank on the same ground set and $\mathcal{C}_o(M') \subseteq \mathcal{C}_o(M)$, then M' is a *relaxation* of M.

The vertices of the *Johnson graph* $J(n, k)$ are the subsets of k elements out of a set of n elements, and each of its edges joins two subsets with $k-1$ elements in the intersection. Therefore, the sparse paving matroids of rank k on n elements are in one-to-one correspondence with the stable sets of the graph $J(n, k)$. A sparse paving matroid is *maximal* if it corresponds to a maximal stable set.

We conclude the section with a discussion about single-element extensions. See [41, Section 7.2] for proofs and a more detailed exposition. Consider sets E, Z with $E \cap Z = \emptyset$. The polymatroid (EZ, g) is an *extension* of the polymatroid (E, f) if $f = (g \backslash Z)$. If (E, f) and (EZ, g) are matroids and $Z = \{z\}$, then (Ez, g) is a *single-element extension* of (E, f). A pair (F_1, F_2) of flats of a matroid (E, f) is *modular* if $f(F_1; F_2 | F_1 \cap F_2) = 0$.

Definition 2.2. A *modular cut* in a matroid M is a family $\mathcal{F}$ of flats satisfying the following properties.

1. If $F \in \mathcal{F}$, then every flat containing F is in $\mathcal{F}$ too.
2. If $F_1, F_2 \in \mathcal{F}$ form a modular pair of flats, then $F_1 \cap F_2 \in \mathcal{F}$.

For every single-element extension (Ez, g) of a matroid $M = (E, f)$, the flats F of M with $g(Fz) = g(F)$ form a modular cut. Conversely, for every modular cut $\mathcal{F}$ in M, there is a single-element extension (Ez, g) of M such that a flat F is in $\mathcal{F}$ if and only if $g(Fz) = g(F)$.

3. Extension properties of matroids and polymatroids

We present in this section a unified approach to several extension properties of matroids and polymatroids that can be found in the literature. In contrast to previous works as [4], our definitions involve iterated extensions. The reader will find a summary about those properties at the end of this section.

Given a polymatroid (E, f) and sets $X, Y \subseteq E$, a *common information* for the pair (X, Y) in (E, f) is a set $Z \subseteq E$ such that $f(Z|X) = f(Z|Y) = 0$ and $f(X; Y|Z) = 0$. A motivation for this concept and its name is found in [19,22]. Not every pair of sets admits a common information, but linear polymatroids can be extended to get one. Indeed, if $(V_x)_{x \in E}$ is a K-linear representation of (E, f), take $z \notin E$, the subspace

$$V_z = \left(\sum_{x \in X} V_x \right) \bigcap \left(\sum_{y \in Y} V_y \right)$$

and the polymatroid (Ez, g) that is K-linearly represented by the collection $(V_x)_{x \in Ez}$. Then (Ez, g) is a K-linear extension of (E, f) and z is a common information for (X, Y) in (Ez, g).

Definition 3.1. For a polymatroid (E, f) and sets $X, Y \subseteq E$, a *common information extension*, or *CI extension* for short, for the pair (X, Y) is an extension (EZ, g) of the polymatroid (E, f) such that Z is a common information for (X, Y) in (EZ, g).

Definition 3.2. We recursively define *k-CI polymatroids*. Every polymatroid is 0-CI. For a positive integer k, a polymatroid is k-CI if, for every pair of subsets of the ground set, it admits a CI extension that is a $(k-1)$-CI polymatroid. A polymatroid satisfies the *common information property*, or it is a *CI polymatroid*, if it is k-CI for every positive integer k.

Definition 3.3. A class of polymatroids satisfies the *common information property* if each of its members does and the CI extensions can be performed inside the class.

Observe that a class of polymatroids satisfies the common information property if and only if, for every polymatroid in the class and every pair of subsets of the ground set, there exists a CI extension inside the class. The next result follows from the discussion opening this section.

Proposition 3.4. *For every field K, the class of K-linearly representable polymatroids satisfies the common information property.*

The previous description of the common information property provides a template to introduce other extension properties. In particular, Definitions 3.2 and 3.3 are easily adapted and we are not going to repeat them. We continue with an extension property for matroids, the *generalized Euclidean property*, which is based on the intersection property with the same name discussed in [4].

Definition 3.5. For a matroid (E, f) and a non-modular pair of flats (F_1, F_2), a *generalized Euclidean extension*, or *GE extension*, for the pair (F_1, F_2) is a matroid (Ez, g) that is a single-element extension of (E, f) satisfying $g(z|F_1) = g(z|F_2) = 0$ and $g(F_1z \cap F_2z) = f(F_1 \cap F_2) + 1$. That is, the extension increases the rank of the intersection of those flats.

And similarly to Proposition 3.4, we have the following result. See [4] for more details.

Proposition 3.6. *For every field K, the class of K-linearly representable matroids satisfies the generalized Euclidean property.*

Different extensions can determine the same extension property. This is the situation for GE extensions and complete Euclidean extensions, which are defined next.

Definition 3.7. For a matroid (E, f) and a pair of flats (F_1, F_2), a *complete Euclidean extension*, or *CE extension*, for the pair (F_1, F_2) is a matroid (EZ, g), which is an extension of (E, f), such that $g(Z|F_1) = g(Z|F_2) = 0$ and (F_1Z, F_2Z) is a modular pair of flats in (EZ, g). Observe that one can take $Z = \emptyset$ if (F_1, F_2) is a modular pair of flats.

Observe that a GE extension can be trivially obtained from a CE extension and, conversely, if there is a CE extension for (F_1, F_2), then there is such an extension that is the result of a sequence of GE extensions. Therefore, they determine the same extension property. Nevertheless, the feasibility of computer-aided explorations may depend on the chosen extension.

A matroid $M = (E, f)$ of rank d satisfies the *Levi's intersection property* if for every $d - 1$ hyperplanes there is an extension of M in which they meet in at least one point. That property, which is discussed in [4], can be used as well to describe an extension property, but it is again equivalent to the generalized Euclidean property. That fact is well known, but we could not find any detailed proof in the literature.

Proposition 3.8. *The extension property for matroids defined from the Levi's intersection property is equivalent to the generalized Euclidean property.*

Proof. Let $M = (E, f)$ be a matroid of rank d. A point in the intersection of $d - 1$ hyperplanes $(H_1, \ldots, H_{d-1})$ of M can be found by a series of CE extensions, beginning with the pair (H_1, H_2), and then the pair formed by the intersection of those hyperplanes and H_3, and so on. For the converse, consider a non-modular pair of flats (F_1, F_2) and take $r_1 = f(F_1)$, $r_2 = f(F_2)$, $s = f(F_1F_2)$, and $t = f(F_1 \cap F_2)$. Since the pair is non-modular, $t < r_1 + r_2 - s$. For $i = 1, 2$, take a basis B_i of F_i and a set B'_i such that $B_iB'_i$ is a basis of the flat spanned by F_1F_2. In addition, take a set B'' such that both $B_1B'_1B''$ and $B_2B'_2B''$ are bases of M. Observe that $|B'_i| = s - r_i$ and $|B''| = d - s$. For every $x \in B'_1B''$, take the hyperplane H_x spanned by $B_1B'_1B'' \setminus x$ while, for every $x \in B'_2$, the hyperplane H_x will be the one spanned by $B_2B'_2B'' \setminus x$. Take $C = B'_1B'_2B''$. Then $(H_x)_{x \in C}$ is a collection of $d + s - r_1 - r_2$ hyperplanes such that each of them contains F_1 or F_2 and their intersection coincides with $F_1 \cap F_2$. Take a basis B of M that contains a basis D of $F_1 \cap F_2$. For every $x \in D$, take the hyperplane H_x spanned by $B \setminus x$. Since $|D| = t$, we have collected up to now $d + s - r_1 - r_2 + t$ hyperplanes, a quantity that is not larger than $d - 1$. Each element in their intersection is in $F_1 \cap F_2$ but it is not in the closure of D, which implies that there is no such element. Finally, the Levi's intersection property implies that M admits a single element extension (Ez, g) such that z is in the intersection of the hyperplanes $(H_x)_{x \in CD}$, but not in the span of $F_1 \cap F_2$. Therefore, (Ez, g) is a GE extension for the pair (F_1, F_2). □

The extension property for the class of algebraic matroids that is defined next is a direct consequence of the Ingleton–Main lemma [30]. The *Dress–Lovász property* is a more restrictive extension property for algebraic matroids that is derived from the generalization of that lemma presented in [21]. The reader is referred to [11] for a description of the Dress–Lovász extension property.

Definition 3.9. For a matroid (E, f) and three lines ℓ_1, ℓ_2, ℓ_3 with $f(\ell_i\ell_j) = 3$ if $1 \le i < j \le 3$ and $f(\ell_1\ell_2\ell_3) = 4$, an *Ingleton–Main extension*, or *IM extension*, for the triple (ℓ_1, ℓ_2, ℓ_3) is a single-element extension (Ez, g) of (E, f) such that $g(z) = 1$ and $g(z|\ell_i) = 0$ for $i = 1, 2, 3$. That is, the three lines intersect in one point in the extension.

Proposition 3.10. *For every field K, the class of K-algebraic matroids satisfies the Ingleton–Main property.*

We present next two extension properties for almost entropic polymatroids. The first one is based on the Ahlswede–Körner lemma [1,2] as stated in [32, Lemma 2]. The second one is based on the copy lemma, which was proved in [20] and was implicitly used before in [46] to find the first known non-Shannon information inequality.

Definition 3.11. For a polymatroid (E, f) and sets $X, Y \subseteq E$, an extension (EZ, g) of (E, f) is an *Ahlswede–Körner extension*, or *AK extension*, for the pair (X, Y) if the following conditions are satisfied.

- $g(Z|X) = 0$
- $g(X'|Z) = g(X'|Y)$ for every $X' \subseteq X$

Definition 3.12. For a polymatroid (E, f) and sets $X_1, X_2, Y \subseteq E$, an extension (EZ, g) of (E, f) is a *copy lemma extension*, or *CL extension*, for (X_1, X_2, Y) if the following conditions are satisfied.

- There is a bijection $\varphi\colon Y \to Z$ that determines an isomorphism between the polymatroids $(X_1Y, (g\backslash A))$ and $(X_1Z, (g\backslash B))$, where $A = EZ \smallsetminus X_1Y$ and $B = EZ \smallsetminus X_1Z$
- $g(Z; X_2Y|X_1) = 0$

Proposition 3.13. *The class of almost entropic polymatroids satisfies both the Ahlswede–Körner and copy lemma properties.*

If a matroid satisfies the generalized Euclidean property, then it satisfies the Ingleton–Main property too, because the latter deals with intersection of lines, that is, flats of rank 2. Every CI-polymatroid is an AK-polymatroid [22]. An additional result on the connection between the common information and Ahlswede–Körner properties is provided by the next proposition, whose elementary proof is given in [22, Proposition III.16].

Proposition 3.14. *Let (EZ, g) be an AK extension of (E, f) for the pair (X, Y). Then $g(Z) = g(X; Y)$. Consequently, (EZ, g) is a CI extension for the pair (X, Y) if and only if $g(Z|Y) = 0$.*

The common information and complete Euclidean properties are very similar. The main difference is that the latter is an extension property of matroids, that is, we require that the extension is a matroid, while the former is an extension property of polymatroids. Folded linear matroids satisfy the common information property but it is not clear that they satisfy the complete Euclidean property, because it could be that the extension derived from a given folded linear representation is not a matroid. Nevertheless, no example of a matroid satisfying the common information property but not the complete Euclidean property is known.

Proposition 3.16 describes a connection between the Ingleton–Main and Ahlswede–Körner extensions. Nevertheless, we have to remember that the first one applies to matroids and the second one to polymatroids. A proof for the following technical result can be found in [22, Proposition III.13].

Lemma 3.15. *Consider a polymatroid (E, f) and subsets $X, Y, U \subseteq E$ such that $f(U|X) = f(U|Y) = 0$. Then $f(U) \le f(X; Y)$. Moreover, if $Z \subseteq E$ is a common information for (X, Y), then $f(U|Z) = 0$.*

Proposition 3.16. *Consider a polymatroid (E, f) and sets $L_0, L_1, L_2 \subseteq E$ with $f(L_i) = 2$, $f(L_iL_j) = 3$ if $i \ne j$, and $f(L_0L_1L_2) = 4$. Let (EZ, g) be an AK extension for the pair (L_1L_2, L_0). Then $g(Z|L_i) = 0$ for each $i = 0, 1, 2$.*

Proof. By Proposition 3.14, $g(Z) = g(L_1L_2; L_0) = 1$ and, by the definition of AK extension, $g(L_1|Z) = g(L_1|L_0) = 1$. Therefore, $g(L_1Z) = g(Z) + g(L_1|Z) = 2$, and hence $g(Z|L_1) = 0$. By symmetry, $g(Z|L_2) = 0$. Since L_0 is a common information for the pair (L_0L_1, L_0L_2) and $g(Z|L_0L_1) = g(Z|L_0L_2) = 0$, by Lemma 3.15 $g(Z|L_0) = 0$. □

The generalized Euclidean property is preserved by taking minors [4], and the same applies to the Ingleton–Main and Dress–Lovász properties [11]. We prove in Propositions 3.17 and 3.18 that both the common information and Ahlswede–Körner properties are preserved by taking minors as well. Since it is obvious that this is the case for deletions, we consider only contractions.

Proposition 3.17. *Consider a polymatroid (E, f) and sets $U \subseteq E$ and $X, Y \subseteq E \smallsetminus U$. Let (EZ, g) be a CI extension of (E, f) for the pair (XU, YU). Then $(EZ \smallsetminus U, (g|U))$ is a CI extension of $(E \smallsetminus U, (f|U))$ for the pair (X, Y).*

Proof. Observe that the minor $(EZ \smallsetminus U, (g|U))$ of (EZ, g) is an extension of $(E \smallsetminus U, (f|U))$. Since $g(U|Z) = 0$ by Lemma 3.15, one can check with a straightforward calculation that $(g|U)(X; Y|Z) = g(XU; YU|Z) = 0$. Finally, $(g|U)(Z|X) = g(Z|XU) = 0$ and, analogously, $(g|U)(Z|Y) = 0$. □

Proposition 3.18. *Consider a polymatroid (E, f) and sets $U \subseteq E$ and $X, Y \subseteq E \smallsetminus U$. Let (EZ, g) be an AK extension of (E, f) for the pair (XU, YU). Then $(EZ \smallsetminus U, (g|U))$ is an AK extension of $(E \smallsetminus U, (f|U))$ for the pair (X, Y).*

Proof. On one hand, $(g|U)(Z|X) = g(Z|XU) = 0$. On the other hand, $g(U|Z) = g(U|YU) = 0$, and hence

$$(g|U)(X'|Z) = g(X'|ZU) = g(X'|Z) = g(X'|YU) = (g|U)(X'|Y)$$

for every $X' \subseteq X$. □

The extension properties we discussed in this section are not useful for matroids of rank 3, because all of them satisfy the common information property and the Ahlswede–Körner property. A proof for this well-known fact can be found in [6, Proposition 3.18]. Nevertheless, there are matroids of rank 3 that are not almost entropic [37] as, for example, the non-Desargues matroid.

Summarizing, we have presented in this section six estension properties. Three of them apply to matroids, namely GE, CE and IM. The first two, which are equivalent, are satisfied by the class of linear matroids, while the third one is satisfied by the class of algebraic matroids. Every GE matroid is IM too. The other three, which are CI, AK, and CL, apply to polymatroids. Only the first two are extensively discussed in this paper. The CI property is closely related CE and it is satisfied by all linear polymatroids. Every CI polymatroid is AK as well. All almost entropic polymatroids satisfy the AK and CL properties.

4. Matroids on eight elements

In this section, we discuss the extension properties of the matroids of rank 4 on 8 elements that are not linearly representable. All of them are sparse paving matroids. We review the known facts and we present a new result about the relaxations of the matroid P_8 that are not folded linear. Namely, we prove that they are not CI matroids. The proof is based on computer explorations, but we prove in a human readable way that two of them do not satisfy the generalized Euclidean property.

We identify the ground set $E = \{0, 1, \dots, 7\}$ of those matroids with the set of vertices $(x, y, z) \in \{0, 1\}^3$ of a 3-dimensional cube. Specifically, each element in E is identified with the vertex corresponding to its binary representation. For instance, 2 is identified to $(0, 1, 0)$ and 6 to $(1, 1, 0)$. The vertices of that cube can be identified with the eight points in the affine space of dimension 3 over the binary field $\mathbf{F}_2$. The circuit-hyperplanes of the sparse paving matroid $AG(3, 2)$ are the 14 planes in that space. Clearly, $AG(3, 2)$ is a maximal sparse paving matroid. It is K-linear if and only if K has characteristic 2 [41].

We begin with a very simple and well known application of extension properties, namely proving that the Vámos matroid is not almost entropic. The Vámos matroid $V = (E, f)$ is the relaxation of $AG(3, 2)$ whose five circuit-hyperplanes are

0123, 0145, 2367, 4567, 2345

Take $L_0 = 01$, $L_1 = 23$, $L_2 = 45$, and $L_3 = 67$ and consider the polymatroid $V_o = (E_o, h)$ with ground set $E_o = \{L_0, L_1, L_2, L_3\}$ and rank function h such that the rank of each singleton equals 2, the sets with two elements have rank 3 except for $h(L_0L_3) = 4$, while the rank of all larger sets is equal to 4. That is, the rank function of V_o is the one induced by the one of V.

Lemma 4.1. *The polymatroid V_o is not 1-AK, and hence it does not satisfy the Ahlswede–Körner property.*

Proof. Assume that there exists an AK extension (E_oZ, g) for the pair (L_1L_2, L_0). Then $g(Z|L_i) = 0$ for each $i = 0, 1, 2$ by Proposition 3.16. Moreover, $g(Z|L_3) = 0$ by Lemma 3.15 because L_3 is a common information for the pair (L_3L_1, L_3L_2) and $g(Z|L_3L_1) = g(Z|L_3L_2) = 0$. Finally, applying Lemma 3.15 again, $1 = g(Z) \le g(L_0; L_3) = 0$, a contradiction. □

As a consequence of the previous proof, there is no AK extension of the Vámos matroid for the pair (L_1L_2, L_0). Therefore, the Vámos matroid is not almost entropic, and hence is not algebraic. The same arguments apply to every matroid presenting the configuration of the circuit-hyperplanes of the Vámos matroid. Specifically, a matroid (E, f) contains the *Vámos configuration* if there are four lines $(L_i)_{0 \le i \le 3}$ such that the polymatroid induced by (E, f) on $E_o = \{L_0, L_1, L_2, L_3\}$ coincides with V_o. Up to isomorphism, there are exactly 39 matroids on eight elements containing the Vámos configuration, which are precisely the matroids on eight elements that do not satisfy the Ingleton inequality [40]. All of them are sparse paving and relaxations of $AG(3, 2)$.

Another relaxation of $AG(3, 2)$ is the sparse-paving matroid L_8, whose circuit-hyperplanes are the six faces of the cube and the two twisted planes 0356, 1247 of $AG(3, 2)$. It is K-linear if and only if $|K| \ge 5$ [41]. The sparse paving matroid L'_8 that is obtained from L_8 by relaxing one of the twisted planes is not linear, but it is folded linear [6]. Moreover, it is algebraic over all fields with positive characteristic [13, Example 35].

The ten circuit-hyperplanes of the sparse paving matroid P_8 are

0246, 1357, 0217, 4617, 2635, 0435, 0637, 0615, 2413, 2457

Following [25], we consider the geometric representation of P_8 that is obtained by rotating half right angle the upper plane 1357 of the cube, as in Fig. 1. It is not difficult to check that P_8 is a maximal sparse paving matroid. By relaxing 2635 from P_8 we obtain the sparse paving matroid $P_{8,1}$. By relaxing 2635 and 1357, we obtain $P'_{8,2}$ while the relaxation of 2635 and 0246 produces the matroid $P''_{8,2}$. Finally, $P_{8,3}$ is the result of relaxing those three circuit-hyperplanes. Observe that P_8, $P_{8,1}$ and $P_{8,3}$ are self-dual (but not identically self-dual) while the dual of $P'_{8,2}$ is isomorphic to $P''_{8,2}$. The matroid P_8 is K-linear if and only if the characteristic of K is different from 2 [41]. By a generalization of the method described in [41, Section 6.4], it was proved in [6] that $P_{8,3}$ is folded linear but not linear while neither $P_{8,1}$, $P'_{8,2}$, nor $P''_{8,2}$ are folded linear. Moreover, $P_{8,3}$ is algebraic over all fields with positive characteristic [34]. It is not known whether $P_{8,1}$, $P'_{8,2}$ and $P''_{8,2}$ are algebraic, almost entropic, or neither.

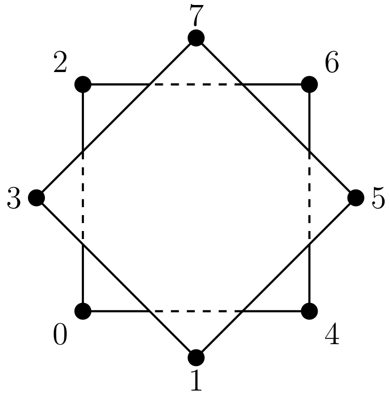

Fig. 1. Geometric representation of P_8.

Table 1
Pairs of sets for which iterated CI extensions at depth 4 do not exist for the matroids $P_{8,1}$, $P'_{8,2}$, and $P''_{8,2}$.

Matroid	(X_1, Y_1)	(X_2, Y_2)	(X_3, Y_3)	(X_4, Y_4)
$P_{8,1}$	01, 56	17, 35	67, 03	13, 57
$P'_{8,2}$	01, 56	17, 35	67, 03	13, 57
$P''_{8,2}$	01, 27	06, 24	67, 14	04, 35

By computer-aided explorations, we checked that those three matroids are not 4-CI. The pairs (X_i, Y_i) for which those common information extensions are not possible are shown in Table 1.

We conclude this section with a human readable proof for the fact that the matroids $P_{8,1}$ and $P'_{8,2}$ do not satisfy the generalized Euclidean property. Let (E, f) be one of the matroids $P_{8,1}$ or $P'_{8,2}$ and let $(Ez_1z_2z_3, g)$ be a triple GE extension of (E, f), where z_1, z_2, z_3 correspond to the non-modular pairs of lines (16, 47), (12, 07), (06, 24), respectively. Since z_1 is on the line 47 and z_2 is on the line 07, the line z_1z_2 is on the plane 047, and hence the lines 04 and z_1z_2 are coplanar. Analogously, the lines 26 and z_1z_2 are coplanar. Observe that z_1 is in the intersection of the planes 0615 and 2457, which is equal to the line z_35. Moreover, the line z_33 is the intersection of the planes 2413 and 0673, which contains z_2. It follows that the lines z_1z_2 and 35 are coplanar. Therefore, the lines 26, 04, z_1z_2 and 35 form the Vámos configuration, which implies that $(Ez_1z_2z_3, g)$ is not 1-GE. We can conclude that (E, f) is not a GE matroid.

5. Matroids with the tic-tac-toe configuration

The tic-tac-toe matroid was introduced as a possible counterexample to prove that the class of algebraic matroids is not closed by duality. It is a sparse paving matroid of rank five on nine elements with eight circuit-hyperplanes. It satisfies the Ingleton inequality but it is not a GE matroid [3]. While it satisfies the Dress–Lovász property [11], it is not known whether it is algebraic or not. Nevertheless, the dual of the tic-tac-toe matroid is not an IM matroid [28], and hence it is not algebraic.

We prove in the following that the tic-tac-toe matroid is not CI and its dual is not AK, and hence the former is not folded linear and the latter is not almost entropic. That applies as well to the sparse paving matroids of rank five on nine elements that contain eight circuit-hyperplanes in the same configuration as the tic-tac-toe matroid. By a computer-aided exploration on the database [43], we found that there are exactly 181 such matroids (see [7]). Once they were determined, we realized that their circuit-hyperplanes can be described in terms of the points and lines on the affine plane over $\mathbf{F}_3$. Moreover, we identified two maximal sparse paving matroids whose relaxations contain all of them.

Let $E = \mathbf{F}_3 \times \mathbf{F}_3$ be the set of points (x, y) on the affine plane over the field $\mathbf{F}_3$. The 12 lines on that affine plane, which are represented in Fig. 2, are partitioned into four sets of three parallel lines. Namely, for $i \in \mathbf{F}_3$,

- the lines A_i with equation $y = i$
- the lines B_i with equation $x = i$,
- the lines C_i with equation $x - y = i$,
- the lines D_i with equation $x + y = i$.

We describe in the following several sparse paving matroids of rank 5 with ground set E that are related to the tic-tac-toe matroid. The 9 circuit-hyperplanes of the matroid T_1^3 are the sets A_iB_j with $(i, j) \in \mathbf{F}_3 \times \mathbf{F}_3$. The *tic-tac-toe matroid*, which is denoted here by T^3, is obtained from T_1^3 by relaxing the circuit-hyperplane A_0B_0. Observe that the relaxation of any other circuit-hyperplane instead of A_0B_0 produces an isomorphic matroid.

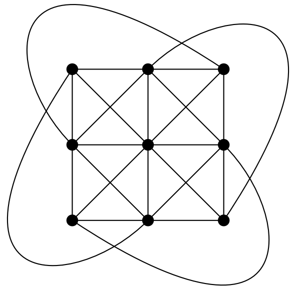

Fig. 2. The affine plane over $\mathbf{F}_3$.

Alfter and Hochstättler [3] proved that the tic-tac-toe matroid T^3 does not satisfy the generalized Euclidean property. We prove next that it does not satisfy the common information property. As a consequence, T^3 is not folded linear.

Proposition 5.1. *The tic-tac-toe matroid T^3 does not satisfy the common information property.*

Proof. Clearly, $Z_0 = \{(0,0)\}$ is a common information for the pair (A_0, B_0) in T^3. Let (EZ, g) be a CI extension of $T^3 = (E, f)$ for the pair (A_{-1}, A_1). Since B_i is a common information for the pair (B_iA_{-1}, B_iA_1) and $g(Z|B_iA_{-1}) = g(Z|B_iA_1) = 0$ it follows by Lemma 3.15 that $g(Z|B_i) = 0$ for each $i \in \mathbf{F}_3$. In particular, Z is a common information for the pair (B_{-1}, B_1) and, by symmetry, $g(Z|A_0) = 0$. Since $g(Z|A_0) = g(Z|B_0) = 0$, we infer that $g(Z|Z_0) = 0$ by Lemma 3.15, and hence $g(Z_0|Z) = 0$ because $g(Z) = g(Z_0)$. That is in contradiction to $g(A_1Z) \neq g(A_1Z_0)$. □

The dual $(T^3)^*$ of the tic-tac-toe matroid is not algebraic because it does not satisfy the Ingleton–Main property [28, Proposition 5]. By using Proposition 3.16, the proof of that result is easily adapted to show that it does not satisfy the Ahlswede–Körner property. Therefore, $(T^3)^*$ is not almost entropic.

Proposition 5.2. *The dual $(T^3)^*$ of the tic-tac-toe matroid does not satisfy the Ahlswede–Körner property.*

Proof. For each pair (i,j) in $\mathbf{F}_3 \times \mathbf{F}_3$, take $L_j^i = \{(i,k),(i,\ell)\}$ with $\{j,k,\ell\} = \mathbf{F}_3$. Observe that $(T^3)^*$ is a sparse paving matroid of rank 4. Its circuit-hyperplanes are the 8 sets $E \smallsetminus A_iB_j$ with $(i,j) \neq \{(0,0)\}$, which coincide with the sets of the form $L_j^iL_j^{i'}$ with $i \neq i'$ other than $L_0^{-1}L_0^1$. Let $(EZ_{-1}Z_1, g)$ be a double AK extension of $(T^3)^*$, where Z_j corresponds to the pair $(L_j^{-1}L_j^1, L_j^0)$. Recall that, since we are dealing with AK extensions, $(EZ_{-1}Z_1, g)$ may not be a matroid. By Propositions 3.14 and 3.16, $g(Z_j) = 1$ and $g(Z_j|L_j^i) = 0$ for each $i \in \mathbf{F}_3$ and $j \in \mathbf{F}_3 \smallsetminus \{0\}$. Take $L = \{Z_{-1}, Z_1\}$. Since $g(Z_{-1}|L_{-1}^{-1}) = g(Z_1|L_1^1) = 0$ and $g(L_1^1|L_{-1}^{-1}) = g(L_1^1)$, it is clear that $g(L) = 2$. If $i \neq i'$, then $g(L_0^i|L_1^{i'}) = g(L_0^i)$, and hence $L \neq L_0^i$ because $g(Z_1|L_1^{i'}) = 0$. In addition, both L and L_0^i are in the span of $\{(i,-1),(i,0),(i,1)\}$, which implies that $g(LL_0^i) = 3$ for each $i \in \mathbf{F}_3$. Therefore, the polymatroid induced by $(EZ_{-1}Z_1, g)$ on $\{L_0^{-1}, L_0^0, L, L_0^1\}$ is isomorphic to V_o and the proof is concluded by Lemma 4.1. □

The previous propositions can be applied as well to the sparse paving matroids on nine elements containing circuit-hyperplanes in the same configuration as T^3 or $(T^3)^*$. That is, the TTT matroids that are described in Definition 5.5 and their duals. We skip the proofs of the next lemma and other similar results in this section because they are elementary but quite cumbersome.

Lemma 5.3. *Let M be a sparse paving matroid on E with $\mathcal{C}_o(T_1^3) \subseteq \mathcal{C}_o(M)$. Then every circuit-hyperplane in $\mathcal{C}_o(M) \smallsetminus \mathcal{C}_o(T_1^3)$ is of the form C_iD_j for some $(i,j) \in \mathbf{F}_3 \times \mathbf{F}_3$.*

Let T_1^m be the sparse paving matroid on E whose 18 circuit-hyperplanes are the sets A_iB_j and C_iD_j with $(i,j) \in \mathbf{F}_3 \times \mathbf{F}_3$. By Lemma 5.3, T_1^m is the only maximal sparse paving matroid that has T_1^3 as a relaxation.

Lemma 5.4. *Let M be a sparse paving matroid on E such that $\mathcal{C}_o(T^3) \subseteq \mathcal{C}_o(M)$. Then there is a unique pair $(i_o, j_o) \in E \smallsetminus A_0B_0$ such that every circuit-hyperplane in $\mathcal{C}_o(M) \smallsetminus \mathcal{C}_o(T^3)$ is either in $\mathcal{C}_o(T_1^m)$ or is equal to $(A_0B_0 \smallsetminus (0,0)) \cup (i_o, j_o)$.*

Consider the sparse paving matroid T_2^3 with the circuit-hyperplanes of T^3 together with $(A_0B_0 \smallsetminus (0,0)) \cup (1,1)$. Taking another point $(i_0, j_o) \in E \smallsetminus A_0B_0$ instead of $(1,1)$ yields a matroid isomorphic to T_2^3.

Definition 5.5. Let M be a sparse paving matroid of rank 5 on E with $\mathcal{C}_o(T^3) \subseteq \mathcal{C}_o(M)$ and $A_oB_0 \notin \mathcal{C}_o(M)$. If $\mathcal{C}_o(M) \subseteq \mathcal{C}_o(T_1^m)$, then M is a *TTT matroid of the first kind*, while M is a *TTT matroid of the second kind* if $(A_0B_0 \smallsetminus (0,0)) \cup (1,1) \in \mathcal{C}_o(M)$.

As a consequence of Lemmas 5.3 and 5.4, every sparse paving matroid of rank five on nine elements that contains eight circuit-hyperplanes in the same configuration as the tic-tac-toe matroid is isomorphic to a TTT matroid, either of the first kind or the second kind.

Neither $C_{-1}D_{-1}$ nor C_1D_{-1} can be circuit-hyperplanes of a TTT matroid of the second kind. Because of that, there is only one maximal sparse paving matroid on E that admits T_2^3 as a relaxation, which we denote by T_2^m. Its 16 circuit-hyperplanes are

- A_iB_j with $(i,j) \neq (0,0)$,
- $(A_0B_0 \setminus (0,0)) \cup (1,1)$, and
- C_iD_j with $(i,j) \neq (-1,-1)$ and $(i,j) \neq (1,-1)$.

Every TTT matroid of the second kind is a relaxation of T_2^m and every TTT matroid of the first kind is a relaxation of T_1^m.

By using the same arguments as in the proofs of Propositions 5.1 and 5.2, no TTT matroid satisfies the common information property, while the Ahlswede–Körner property is not satisfied by any dual of a TTT matroid.

By applying the method described in [41, Section 6.4], one can check that the matroid T_1^m is linearly representable, but only over fields of characteristic 3. A representation over $\mathbf{F}_3$ is given by the following matrix, whose columns are indexed by the elements in E in the order $(-1,-1), (-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0), (1,1)$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

The matroid T_1^3 is linear over fields of every characteristic. A linear representation is given in [6, Section 5].

The TTT matroids of the second kind that contain the circuit-hyperplanes

$$C_0D_{-1}, C_0D_0, C_0D_1, A_1B_1, (A_0B_0 \setminus (0,0)) \cup (1,1) \tag{1}$$

are not almost entropic. Indeed, let M be such a matroid. By removing $(1,1)$ from each of the sets in (1), we obtain five circuit-hyperplanes of the minor of M that results from the contraction of $(1,1)$. It is easy to check that they form a Vámos configuration. For every TTT matroid of the second kind in that situation, the contraction of $(1,1)$ yields a sparse paving matroid of rank 4 on 8 elements whose 9 circuit-hyperplanes are obtained by removing $(1,1)$ from each of the sets

$$C_0D_{-1}, C_0D_0, C_0D_1, A_1B_1, (A_0B_0 \setminus (0,0)) \cup (1,1), A_1B_{-1}, A_1B_0, A_0B_1, A_{-1}B_1$$

and hence that minor is isomorphic to the matroid with tag 1509 in the database by Royle and Mayhew [43]. There are exactly 10 TTT matroids of the second kind in that situation, which are specified in [7]. The matroid T_2^m is one of them.

The representability of matroids on nine elements by Frobenius-flocks over fields of characteristic 2 was exhaustively explored by Bollen [11]. By matching the results of that exploration with our list of TTT matroids, we found that at least 62 of them are not Frobenius-flock representable in characteristic 2, and hence they are not algebraic over fields of characteristic 2. Those TTT matroids have at least 12 circuit-hyperplanes. They are listed in the extended version of this work [7]. In particular, for all fields of positive characteristic, to determine whether the tic-tac-toe matroid T^3 is algebraic or not remains an open problem.

Acknowledgment

We thank the anonymous reviewer for carefully revising our paper and providing valuable suggestions for its improvement.

Data availability

We shared the links to the code and databases in the paper.

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