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Cooperative TU-games: Dominance, stable sets, and the core revisited[☆]

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ABSTRACT

Stable sets are introduced by von Neumann and Morgenstern (1944) as "the solution" of a cooperative game. Later on, Gillies (1953) defines the core of the game. Both notions can be established in terms of dominance. It is well known that the core may be an empty set, whereas stable sets may fail to exist, or may produce different proposals. We provide a new dominance relation so that the stable set obtained when applying this notion (the δ-stable set) always exists, it is unique, and it coincides with the core of the cooperative game, whenever the core is not empty. We apply this concept to some particular classes of TU-games having typically an empty core: voting (majority) games, minimum cost spanning trees games with revenue, controlled capacitated networks, or m-sequencing games.

1. Introduction

Given a cooperative TU-game, it is usually assumed that the grand coalition N of all involved players is formed, and the question comes down to how to distribute the amount they obtain by cooperation among the players. The ways of distributing the obtained worth, $v(N)$, are called *solution concepts*, and they are usually based on different interpretations of fairness. For example, one possible criterion is stability: players should not have an incentive to ask for a different share of the worth. The *core* and the (von Neumann-Morgenstern) *stable set* are two central notions of solution for cooperative TU-games. Both notions are multivalued solutions, so they can be considered as a pool where to look for solving the situation (game).

It is noteworthy that the stable set and the core can be defined through a *dominance relation* on the set of possible shares of $v(N)$. This dominance is defined in terms of stability of the grand coalition: *a sharing x of $v(N)$ is dominated if a sub-coalition of individuals $S \subseteq N$ exists such that their members can obtain a worth $v(S)$ greater than the amount assigned by x to the players in that coalition.* Then, the core in superadditive games (see Section 2 for formal definitions) is the set of undominated shares, whereas a stable set of a game is defined as a subset of imputations V satisfying *internal stability* (no element in V is dominated by other element in V) and *external stability* (each element outside V is dominated by some element in V).

Nonetheless, both solutions present some shortcomings:

- The core may be the empty set (leaving the problem unsolved).
- The stable set may not exist (leaving the problem unsolved).
- Several (different) stable sets may exist.
- The core and the stable set may provide very different pools where to look for a particular share of the worth. In Lucas (1992) a 5-players cooperative TU-game is shown such that the core is non-empty, there exists a unique stable set, but both proposals are different from each other.
  However, for superadditive games, the core is a subset of every stable set. Moreover, if the core is a stable set, it is the unique stable set (Peters, 2008).

This paper mainly addresses the potential emptiness of the core. In doing so, we introduce the δ-stable set by using an alternative dominance relation. This dominance relation is based on the number of agents (the size of a coalition) that would accept a particular distribution of the total worth $v(N)$, ensuring that each sub-coalition also accepts this distribution. We figure out that a unique, non-empty set always exists that satisfies these conditions and coincides with the core whenever the core is not empty. In this way, the δ-stable set can be seen as an extension of the core.

To illustrate our concept, we use well-known examples of cooperative situations where the core may be empty. We analyze voting (majority) games, a situation in which the core is tipically empty (unless a veto player exists), and we focus particularly on the case of the

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minimum cost spanning tree with revenue, as this scenario often results in an empty core, even in simplified network structures. By treating the δ-stable set as the set of reasonable payoffs in the cost-revenue game, we avoid the issue of core emptiness and establish a core assignment whenever possible. Additionally, we also discuss the cases of controlled capacitated networks, or m-sequencing games.

The rest of the paper is organized as follows. Section 2 contains the basic definitions used throughout the paper: TU-games, *core*, *stable set*, among others. Section 3 presents our approach to stability, by defining a new dominance relation. We show some general properties and relate the δ-stable set with the core of a cooperative game. Section 4 is devoted to analyze the properties that the δ-stable set satisfies, comparing with some satisfied by the core. In Section 5 we illustrate our concept with some well-known cooperative situations. Some final comments close the paper in Section 6.

2. Preliminaries

This section primarily introduces the definitions and notation established in Owen (1982) and Peters (2008).

Definition 1. A **cooperative game** with transferable utility, or TU-game, is a pair (N, v), where $N = \{1, 2, \dots, n\}$, with $n \in \mathbb{N}$, is the **set of players**, and v is a function assigning to each **coalition** S, i.e., to each subset $S \subseteq N$ a real number $v(S)$, such that $v(\emptyset) = 0$. The function v is called the **characteristic function** and $v(S)$ is called the worth of S. The coalition N is called the **grand coalition**.

Definition 2. A TU-game (N, v) is **essential** if $v(N) \geqslant \sum_{i\in N} v(i)$. It is **inessential** otherwise.

Definition 3. A TU-game (N, v) is **superadditive** if

$$\text{for all } S, T \subseteq N,\ S \cap T = \emptyset \quad \Rightarrow \quad v(S \cup T) \geqslant v(S) + v(T)$$

Many games v derived from practical situations are superadditive, thus this paper focuses exclusively on this type of TU-games.

Definition 4. Let (N, v) be a TU-game. A vector $x \in \mathbb{R}^n$ is called an **imputation** if

(1) x is **individually rational**, i.e. $x_i \geqslant v(i)$, for all $i \in N$; and
(2) x is **efficient**, i.e. $x(N) = v(N)$.

The set of imputations of a game (N, v) is denoted by $\mathcal{X}(v)$. An element $x \in \mathcal{X}(v)$ represents a payoff distribution of the total worth $v(N)$ of the grand coalition N, ensuring that each player i receives a payoff x_i that is at least as much as they could obtain independently.

Notation. Given a vector $x \in \mathbb{R}^n$ and a subset $S \subseteq N$, we denote $x(S) = \sum_{i\in S} x_i$.

Asking for the TU-game be essential is equivalent to the non-emptiness of the set of imputations $\mathcal{X}(v)$ of the game.[1] Therefore, it is possible to compare these imputations and seeking those that are, in some sense, more favorable.

Definition 5. Let (N, v) be an essential TU-game. Let $x, y \in \mathcal{X}(v)$, and $S \subseteq N$, $S \neq \emptyset$, a coalition. Then x **dominates** y **via coalition** S, denoted by $x \operatorname{dom}_S y$, if

(1) $x_i > y_i$, for all $i \in S$; and
(2) $x(S) \leqslant v(S)$.

For $x, y \in \mathcal{X}(v)$, x is said to **dominate** y, denoted by $x \operatorname{dom} y$, if there is a coalition S such that $x \operatorname{dom}_S y$.

In the above definition, (1) establishes that the payoff distribution x is better than y for all members $i \in S$, since all of them receive a greater payoff. Moreover, (2) indicates that the payoffs $(x_i)_{i\in S}$ are attainable for the members of S by cooperation. In other words, agents in S prefer the payoffs from x to those from y, and they can threaten to leave the grand coalition if y is used because the payoff they can obtain on their own is at least as large as the allocation they receive under x.

Given a TU-game (N, v), for any coalition $S \subseteq N$, the set $D(S)$ consists of the imputations which are dominated via coalition S,

$$D(S) = \left\{z \in \mathcal{X}(v) : \text{there exists } x \in \mathcal{X}(v) \text{ such that } x \operatorname{dom}_S z\right\}$$

Against each $z \in D(S)$, the players of S can object successfully, since they can obtain a better worth than the one provided by z. Note that $D(N) = \emptyset$. We call $x \in \mathcal{X}(v)$ ***undominated*** if there is not $z \in \mathcal{X}(v)$ and $S \subseteq N$ such that z dominates x via S. Let $U(N, v) = \mathcal{X}(v) \setminus \bigcup_{S\subseteq N} D(S)$ denote the set of ***undominated*** imputations.

By using the above dominance relation on imputations, von Neumann and Morgenstern (1944) introduce the notion of *stable* set as a "solution" for cooperative games.

Definition 6. Let (N, v) be a TU-game. A subset $V \subseteq \mathcal{X}(v)$ is a **stable set** if:

(a) **Internal stability**: No payoff vector in the stable set V is *dominated* by another payoff vector in this set

$$x, y \in V \quad \Rightarrow \quad x \operatorname{dom} y \text{ never holds}$$

(b) **External stability**: All payoff vectors outside the set V are *dominated* by at least one payoff vector in the stable set V

$$z \notin V \quad \Rightarrow \quad \text{there exists } x \in V \text{ such that } x \operatorname{dom} z$$

Since von Neumann and Morgenstern defined the stable set, its general existence has been an important problem in game theory. The problem was eventually solved in a negative way: in general, stable sets may not exist (Lucas, 1969). Moreover, when they do exist, it is possible that several different stable sets exist. Although the general existence of stable sets is denied, they offer valuable insights into a wide range of economic, political, and social issues (Wako and Muto, 2009).

Another solution concept that is widely used in the cooperative game theory is the *core* of the game, which is first introduced by Gillies (1953).

Definition 7. The **core** of a TU-game (N, v) is the set

$$Core(N, v) = \{x \in \mathcal{X}(v) \text{ such that: } x(S) \geqslant v(S) \text{ for all } S \subseteq N\}\,.$$

If $x \in Core(N, v)$, then no coalition S has an incentive to deviate when x is the proposed payoff distribution for the grand coalition N, since the total amount allocated to S, denoted by $x(S)$, is not less than the amount $v(S)$ that the members of S could secure on their own.

The following result characterizes the core of a TU-game in terms of the dominance relation.

Lemma 1 (Peters, 2008)**.** *Let (N, v) be a superadditive TU-game. Then,*

$$Core(N, v) = U(N, v) = \{x \in \mathcal{X}(v) : z \operatorname{dom} x \text{ for no } z \in \mathcal{X}(v)\}\,.$$

From the above result, the core of a superadditive TU-game (N, v) is contained in any stable set V, if the latter exists, since no imputation from $Core(N, v)$ can be dominated. On the other hand, the same argument shows that $Core(N, v)$ satisfies the internal stability condition in

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[1] The set of imputations of essential TU-games is the convex hull of the points $F_1, F_2, \dots, F_n$, where for $k = 1, 2, \dots, n$

$$F_i^k = \begin{cases} v(k) & \text{if } i \neq k \\ v(N) - \sum_{k\neq i} v(k) & \text{otherwise} \end{cases}$$

(see, for instance, Peters, 2008).

Definition 6. When the core of a game satisfies the external stability in Definition 6, it has very strong stability and it is called the *stable core*. If it exists, the stable core is the unique stable set of the game (Wako and Muto, 2009).

The core of a TU-game always exists. Nevertheless, it can be the empty set, so no solution is proposed in this case. As mentioned in Serrano (2009): “A game with an empty core is to be understood as a situation of strong instability, as any payoffs proposed to the grand coalition are vulnerable to coalitional blocking”.

3. A new dominance relation

Lucas and Rabie (1982) describe superadditive TU-games such that there is no stable set, and for which the core is the empty set. Despite these discourage results, the core is a central solution concept of TU-games, and the von Neumann–Morgenstern solution is a compelling proposal. Nevertheless, it is of interest to study some alternative types of stability whenever core or stable set fail to provide a solution of the problem. In the literature on cooperative game theory we can find the notions of the *nucleolus*, which belongs to the core whenever the core is not empty, the *kernel*, the *bargaining set* (by using a coalition structure), the *Weber set*, or the *Shapley value*, which belongs to the core whenever the game is convex (Owen, 1982; Peters, 2008).

Thus, we are interested in demanding less strong types of stability than the one required by the core, expecting to obtain less precise yet more general results. And we do it by defining an alternative notion of dominance among imputations. First, we define an index associated to each imputation.

Definition 8. Let (N, v) be a TU-game. To each imputation $x \in \mathcal{X}(v)$ we associate the number

$$s(x) = \max\left\{|S|, S \subseteq N : \text{ for all } R \subseteq S,\ x(R) \geqslant v(R),\ x(S) = v(S)\right\}.$$

We refer to this number as the **hierarchical coalitional rationality** (*HCR*) index of the imputation x in the TU-game (N, v).[2]

Definition 9. Let (N, v) be a TU-game. Given $x, y \in \mathcal{X}(v)$, we say that x **δ-dominates** y (denoted by $x \rhd y$) if $s(x) > s(y)$.

Remark 1. The *HCR* index $s(x)$ determines the maximum number of individuals (the size of the coalition) so that no sub-coalition objects against the distribution of the net revenue provided by x and is feasible for S. Obviously, $s(x) \in \{0, 1, \ldots, n\}$. If the game is essential, that is $v(N) \geqslant \sum_{i \in N} v(i)$, we can always consider the imputation $\bar{x} \in \mathcal{X}(v)$

$$\bar{x} = \left(v(1), v(2), \ldots, v(N) - \sum_{i=1}^{n-1} v(i)\right)$$

and there are coalitions of at least one individual fulfilling the condition in Definition 9, so that $s(\bar{x}) \geqslant 1$. Note also that if $s(x^*) = n$ for some $x^* \in \mathcal{X}(n)$, this implies that x^* belongs to the core of the TU game (N, v). Therefore, the existence of such an imputation x^* is equivalent to the core being non-empty.

Now, by using the δ-dominance, we define our stability notion in the usual way.

Definition 10. Let (N, v) be a TU-game. A subset $V \subseteq \mathcal{X}(v)$ is a **δ-stable set** if:

(a) **Internal stability**: For all $x, y \in V$, no $x \rhd y$; and
(b) **External stability**: For all $z \notin V$, there is $x \in V$ such that $x \rhd z$.

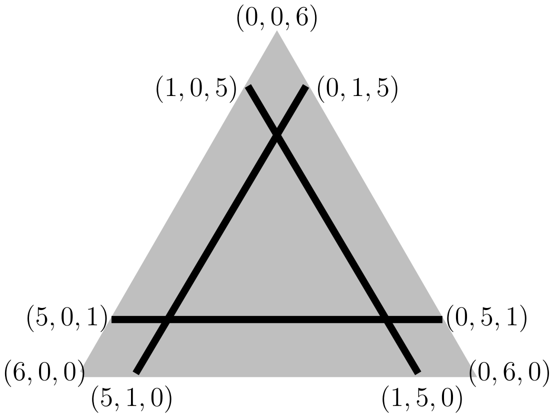

Fig. 1. The δ-stable set of the game in Example 1 is represented by the black lines. The gray triangle corresponds to the set of imputations of the TU-game.

The following example illustrates the idea of δ-stable set.

Example 1. Let $N = \{1, 2, 3\}$, and the TU-game defined by:
$v(i) = 0$, for all $i \in N$; $v(i, j) = 5$, for all $i, j \in N$, $i \neq j$; $v(N) = 6$.
Then, the set of imputations is

$$\mathcal{X}(v) = \left\{ x \in \mathbb{R}^3_+ : \sum_{i=1}^{3} x_i = 6 \right\}$$

The core of this game is empty, $Core(N, v) = \emptyset$, since

$$\forall\, i \neq j,\ x_i + x_j \geqslant 5\ \Rightarrow 2x_1 + 2x_2 + 2x_3 \geqslant 15$$

contradicting $x_1 + x_2 + x_3 = 6$.

On the other hand, several stable sets exist. For instance the following sets are stable in the von Neumann–Morgenstern sense:

$$V^1 = \{(3,3,0), (3,0,3), (0,3,3)\}$$
$$V^2 = \left\{\left(x_1, x_2, 1\right) : x_1 + x_2 = 5,\ x_1 \geqslant 0,\ x_2 \geqslant 0\right\}$$
$$V^3 = \left\{\left(x_1, 1, x_3\right) : x_1 + x_3 = 5,\ x_1 \geqslant 0,\ x_3 \geqslant 0\right\}$$
$$V^4 = \left\{\left(1, x_2, x_3\right) : x_2 + x_3 = 5,\ x_2 \geqslant 0,\ x_3 \geqslant 0\right\}$$
$$V^5 = \left\{\left(x_1, x_2, x_3\right) \in \mathcal{X}(v) : x_1 = x_2 \geqslant \frac{5}{2}, \text{ or } x_1 = x_3 \geqslant \frac{5}{2}, \text{ or } x_2 = x_3 \geqslant \frac{5}{2}\right\}$$

But there is only one δ-stable set of this game:

$$V^\delta = \left\{x \in \mathcal{X}(v) : x_i + x_j = 5 \quad \text{for some } i, j \in N,\ i \neq j\right\}$$

that coincides with the union of V^2, V^3, V^4 (see Theorem 4 regarding this question). Fig. 1 illustrates this example.

The idea behind the δ-stable set in this example relies on the notion of a *minimal right* for non-supporting agents. When cooperating, all three agents can collectively achieve a total worth of 6 units. However, any coalition of two agents can earn only 5 units. Consequently, there is 1 unit that cannot be claimed by any two-agent coalition, and this surplus should be allocated to the remaining agent. Note that the von Neumann–Morgenstern solution V^1 fails to fulfill this requirement.

Unlike stable sets, which may or may not exist (Lucas, 1969), our next result states the existence and uniqueness of δ-stable sets.

Theorem 1. *For each essential TU-game there is always a unique non-empty δ-stable set.*

Proof. Let us consider the set of imputations maximizing the *HCR* index $s(x)$

$$V = \{x \in \mathcal{X}(v) : s(x) \geqslant s(y), \text{ for all } y \in \mathcal{X}(v)\}.$$

This set is non-empty, since $\mathcal{X}(v) \neq \emptyset$ and the *HCR* index $s(x)$ only achieves a finite number of possible values, $s(x) \in \{0, 1, \ldots, n\}$. We first show that V is a δ-stable set.

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[2] We thank an anonymous referee for suggesting this name for the index, that allows to interpret it as a measure of the *degree of hierarchical coalitional rationality* of the imputation x.

(a) Internal stability: If $x, y \in V$, then $s(x) = s(y)$ and no imputation dominates the other.
(b) External stability: If $z \notin V$, then z does not maximize the *HCR* index s, so $s(x) > s(z)$, for all $x \in V$, and z is dominated by each element in V.

Let us suppose now that $W \neq V$ is also a δ-stable set. Then:

1. If there is $x \in V$ such that $x \notin W$, as W is δ-stable, $y \in W$ exists such that $y \rhd x$, so $s(y) > s(x)$, contradicting the definition of set V. Therefore, $V \subseteq W$.
2. If there is $y \in W$ such that $y \notin V$, as V is δ-stable, $x \in V$ exists such that $x \rhd y$, contradicting internal stability of W, since $x, y \in W$.

Therefore, $V = W$ and there is a unique δ-stable set. ■

So, the above result allows us to define *the δ-stable set* of an essential TU-game (N, v), that we denote by $St^{\delta}(N, v)$. An immediate, though interesting, consequence is that the δ-stable set coincides with the core of the game, whenever the core is not empty.

Theorem 2. *In each essential TU-game (N, v) having a non-empty core the δ-stable set coincides with the core.*

Proof. It is obvious that the set $Core(N, v)$ is δ-stable, whenever non-empty. Moreover, as mentioned, $x \in Core(N, v)$ if, and only if, $s(x) = n$. From the uniqueness obtained in Theorem 1 this is the unique stable set. ■

Remark 2. Although we assume throughout the paper that the TU-games are superadditive, it is noteworthy that, by Theorem 1, we only need the TU-game to be essential in order to ensure existence (and unicity) of the δ-stable set; that is, we do not need superadditivity to define our solution concept.

4. Axiomatic analysis

The axiomatic study of solution concepts in cooperative games provides a normative foundation that elucidates their mathematical structure and economic relevance. Within the framework of TU-games, the *core* emerges as a pivotal solution concept, capturing the idea of stability through collective rationality. Traditional characterizations of the core typically rely on axioms such as efficiency, coalitional rationality, and various forms of consistency. Prominent axiomatic approaches can be found in Davis and Maschler (1965), or Peleg and Sudholter (2003). A refined perspective on consistency was introduced by Tadenuma (1992), who employed reduced games in the sense of Moulin (1985). His formulation requires that any allocation in the core of the original game yields, upon removal of a player, an allocation in the core of the corresponding reduced game. This approach captures a dynamic notion of stability under the exclusion of players. Tadenuma's main result demonstrates that efficiency and reduced-game consistency alone suffice to characterize the core. As aforementioned, consistency expresses the idea that if a certain allocation is selected as a solution in a game, then when one player is removed and assigned their payoff, the remaining players' allocation should still be a solution to the appropriately reduced game. This notion captures a form of internal coherence of the solution when composition of the game changes.

Reduced Game à la Moulin: Let (N, v) be a TU game and suppose $x \in \mathbb{R}^N$ is a payoff vector. For each $T \subseteq N$, the *reduced game* with respect to x, denoted by v_T^x, is defined for every coalition $S \subseteq N \setminus T$ as:

$$v_T^x(S) = v(S \cup T) - x(T)$$

Consistency à la Moulin: A solution F satisfies *(complement) consistency*[3] if for every game (N, v) and every $x \in F(N, v)$, it holds that:

$$x_{N \setminus T} \in F(N \setminus T, v_T^x), \quad \forall T \subseteq N$$

where $x_{N \setminus T}$ denotes the restriction of x to the set $N \setminus T$.

Consistency is closely linked to coalitional stability, as this axiom becomes incompatible with efficiency and individual rationality in cases where the core is empty, as demonstrated by the following result.

Theorem 3. *There is no solution defined in the family of superadditive TU-games satisfying efficiency, non-emptiness, individual rationality and consistency.*

Proof. Let $N = \{1, 2, 3\}$ and the characteristic function defined by:

$$v(\emptyset) = 0,\ v(\{1\}) = 0.4,\ v(\{2\}) = 0,\ v(\{3\}) = 0,$$

$$v(\{1,2\}) = 1,\ v(\{1,3\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,2,3\}) = 1.4$$

Let F be a non-empty solution concept that satisfies individual rationality and efficiency, i.e., for every $x \in F(N, v)$, it holds that $x(N) = v(N)$ and $x_i \geqslant v(\{i\})$ for all $i \in N$. Let (N, v) be a TU-game and consider an allocation $x \in F(N, v)$. By efficiency and individual rationality, the solution can be expressed as:

$$x = (0.4 + \varepsilon, \alpha, 1 - \varepsilon - \alpha) \qquad \varepsilon \geqslant 0,\ \alpha \geqslant 0,\ \varepsilon + \alpha \leqslant 1$$

Let us suppose that F also satisfies consistency and consider the following cases:

1. Player 2 leaves the group. Then,

   $$v_2^x(\emptyset) = 0,\ v_2^x(\{1\}) = 1 - \alpha,\ v_2^x(\{3\}) = 1 - \alpha,\ v_2^x(\{1,3\}) = 1.4 - \alpha$$

   If $(0.4 + \varepsilon, 1 - \varepsilon - \alpha) \in F(\{1, 3\})$, individual rationality implies:

   $$0.4 + \varepsilon \geqslant 1 - \alpha \text{ and } 1 - \varepsilon - \alpha \geqslant 1 - \alpha \quad \Rightarrow \quad \varepsilon = 0$$

   Then the solution is

   $$x = (0.4, \alpha, 1 - \alpha) \qquad 0 \leqslant \alpha \leqslant 1$$

2. Player 1 leaves the group. Then,

   $$v_1^x(\emptyset) = 0,\ v_1^x(\{2\}) = 0.6,\ v_1^x(\{3\}) = 0.6,\ v_1^x(\{2,3\}) = 1$$

   If $(\alpha, 1 - \alpha) \in F(\{2, 3\})$, individual rationality implies:

   $$\alpha \geqslant 0.6 \text{ and } 1 - \alpha \geqslant 0.6\ (\alpha \leqslant 0.4)$$

   a contradiction.

Therefore, a solution fulfilling the above axioms cannot exist. ■

Remark 3. Davis and Maschler (1965) introduce an alternative notion of consistency, which involves a different definition of the reduced game. The example presented in the preceding proof can also be used to establish that, under Davis-Maschler's formulation of consistency, an impossibility result still arises.

Since our solution satisfies efficiency, individual rationality, and it is always non-empty, it cannot satisfy consistency. However, as shown after Theorem 4, consistency holds for the corresponding sub-problems when considering coalitions that maximize the index s(x).

Given that our analysis focuses on settings in which the core may be empty, it might be that for each feasible allocation, some players raise objections while others accept it. The following result provides an alternative interpretation of the δ-stable set: it coincides, *in a certain sense*, with the union of the core of subgames associated with coalitions that maximize the *HCR* index $s(x)$.

To formally establish this result, we introduce some auxiliary notation. The subgame induced by a coalition $S \subseteq N$ is denoted by (S, v_S), where the characteristic function v_S is defined for all subsets $R \subseteq S$ as $v_S(R) = v(R)$.

Given a TU-game (N, v),

- $s^* = \max\{s(x) : x \in \mathcal{X}(v)\}$

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[3] The name of this property is borrowed from Thomson (2011).

- $\Lambda(v) \;=\; \{S \subseteq N,\ |S| = s^* : \exists x \in \mathcal{X}(v)$ such that $\forall R \subseteq S,\ x(R) \geqslant v(R),\ x(S) = v(S)\}$
- for $x \in \mathcal{X}(v)$ and $S \subseteq N$, x_S is the projection of x in S
- for $S \subseteq N$, $\overline{Core}(S, v) = \left\{x \in \mathcal{X}(v) :\ x_S \in Core(S, v_S)\right\}$

Theorem 4. *In each essential TU-game (N, v)*

$$St^{\delta}(N, v) = \bigcup_{S \in \Lambda(v)} \overline{Core}\,(S, v)$$

Proof. If $x \in St^{\delta}(N, v)$ then there is some coalition S maximizing the *HCR* index $s(x)$ such that for all $R \subseteq S, x(R) \geqslant v(R)$, and $x(S) = v(S)$. That is, $x_S \in Core\left(S, v_S\right)$ for some $S \in \Lambda(v)$, so $x \in \overline{Core}\,(S, v)$.

Conversely, let $x \in \overline{Core}\,(S, v)$, for some coalition $S \in \Lambda(v)$. If we suppose $x \notin St^{\delta}(N, v)$ then there is some $y \in St^{\delta}(N, v)$ that dominates x, so $s(y) > s(x)$, which is not possible. ■

Building on the previous theorem, for every coalition S contained in $\Lambda(v)$, the consistency property holds for the players in S since the core, whenever non-empty, satisfies the consistency axiom.

5. Cooperative *TU*-games with empty core

5.1. Voting (majority) games

A classical example of TU-games with an empty core is that of *voting games*. A TU-game (N, v) is a ***voting game*** when:

- The valuation function takes only two values: 1 for the winning coalitions, 0 otherwise.
- v satisfies unanimity: $v(N) = 1$.
- v satisfies monotonicity: $S \subseteq T \Rightarrow v(S) \leqslant v(T)$.

It is noteworthy that these games are usually called *simple games*. It is well-known that the core of a voting game is nonempty if and only if there exists a player with veto power (see, for instance, Peleg and Sudholter, 2003). Therefore, "the core is not a useful tool for analyzing voting situations with no veto player. In simple majority voting games, no player has veto power, and thus, the core is empty" (Wako and Muto, 2009). A simple three-person game illustrates this fact.

Example 2. Let $N = \{1, 2, 3\}$ and the majority voting game defined by $v(S) = 1$ if $|S| \geqslant 2$, $v(S) = 0$, otherwise. The core of this game is empty and the different stable sets are (von Neumann and Morgenstern, 1944):

$$V^0 = \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\}$$

$$V^1_{\alpha} = \left\{(\alpha, x_2, x_3) \in \mathbb{R}^3_+ : x_2 + x_3 = 1 - \alpha\right\} \qquad 0 \leqslant \alpha < \tfrac{1}{2}$$

$$V^2_{\alpha} = \left\{(x_1, \alpha, x_3) \in \mathbb{R}^3_+ : x_1 + x_3 = 1 - \alpha\right\} \qquad 0 \leqslant \alpha < \tfrac{1}{2}$$

$$V^3_{\alpha} = \left\{(x_1, x_2, \alpha) \in \mathbb{R}^3_+ : x_1 + x_2 = 1 - \alpha\right\} \qquad 0 \leqslant \alpha < \tfrac{1}{2}$$

The first three-point stable set V^0 indicates that a two-person coalition is formed, and that agents in the coalition share equally the outcome obtained. The other three types of stable sets, V^i_{α}, $i = 1, 2, 3$, $\alpha \in \left[0, \frac{1}{2}\right)$, indicates that agent i gets a fixed amount α and other agents negotiate for how to allocate the rest $1 - \alpha$. The first stable set V^0 is called a *symmetric* (or objective) stable set, while the latter types are called *discriminatory* stable sets (Wako and Muto, 2009).

As we know, there always exists only one δ-stable set, which in this case is the subset of allocations

$$St^{\delta}(N, v) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_i + x_j = 1,\ i \neq j,\ \ i, j \in \{1, 2, 3\}\right\}$$

Note that contrary to some stable sets, in this problem the δ-stable set is symmetric, in the sense that if an allocation $\left(x_1, x_2, x_3\right) \in V^*$, then $\left(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}\right) \in V^*$, for any permutation π. The only symmetric stable set is V^0, which is included in $St^{\delta}(N, v)$.

5.2. Minimum cost spanning trees games with revenue

Our next application deals with minimum spanning tree problems (*mcst* in what follows). We consider a scenario where individuals situated in various locations seek connection to a source to access a good or service, with each link between individuals or between an individual and the source incurring a specific fixed cost. Once the optimal tree is identified, an essential consideration is the allocation of the minimum cost among the individuals, which has been explored in existing literature (e.g., Bogomolnaia and Moulin (2010) and Bergantiños and Vidal-Puga (2021)), proving that these problems always have non-empty cores. Additionally, Estévez-Fernández and Reijnierse (2014) investigate scenarios involving a general service, focusing on the distribution of net revenues generated by cooperation among agents.[4] They demonstrate that the core of the resulting game may be empty, even when only two cost values (low and high) are present. The following example shows this situation.

Example 3 (*Estévez-Fernández and Reijnierse, 2014*)**.** Let us consider the *mcst* problem with revenue represented in Fig. 2.

The corresponding cost-revenue game is given by

$$v(S) = \begin{cases} 1 & if\ \{1,2,4\} \subseteq S, \{1,3,5\} \subseteq S, or\ \{2,3,6\} \subseteq S \\ 0 & otherwise \end{cases}$$

Hence, $Core\,(N, v) = \emptyset$.

Now we obtain the δ-stable set. Since the core of (N, v) is empty, it is not possible to find a vector $x \in \mathcal{X}(v)$ such that $s(x) = 6$. By considering coalition $S = \{2, 3, 4, 5, 6\}$, the payoff vector $x = (0, 1/3, 1/3, 0, 0, 1/3)$ satisfies $x(R) \geqslant v(R)$, for all $R \subseteq S$. So, $s(x) = 5$ and this is the maximum value of the *HCR* index $s(x)$. The δ-stable set is:

$$St^{\delta}(N, v) = \left\{x \in \mathbb{R}^6_+ : x_1 + x_2 + x_4 = 1,\ or\ x_1 + x_3 + x_5 = 1,\ or\ x_2 + x_3 + x_6 = 1\right\}$$

The following subsets are (discriminatory) von Neumann–Morgenstern stable sets:

$$V^1 = \left\{\left(x_1, x_2, 0, x_4, 0, 0\right) :\ x_1 + x_2 + x_4 = 1,\ x_i \geqslant 0\right\}.$$

$$V^2 = \left\{\left(x_1, 0, x_3, 0, x_5, 0\right) :\ x_1 + x_3 + x_5 = 1,\ x_i \geqslant 0\right\}.$$

$$V^3 = \left\{\left(0, x_2, x_3, 0, 0, x_6\right) :\ x_2 + x_3 + x_6 = 1,\ x_i \geqslant 0\right\}.$$

Thus the δ-stable set consists of the subset of $\mathcal{X}(v)$ containing all these possible payoffs, that can be considered "reasonable" distributions of the net revenue. Note that, as established in Theorem 4, the payoffs defined by the following conditions (1), (2) and (3) coincide with the union of core of the sub-games involving the coalitions that provide the maximum value of $s(x)$:

(1) $\overline{Core}\left(S_1, v\right) = \left\{x \in \mathcal{X}(v) : x_1 + x_2 + x_4 = 1\right\}$, $S_1 = \{1, 2, 4, 5, 6\}$.
(2) $\overline{Core}\left(S_2, v\right) = \left\{x \in \mathcal{X}(v) : x_1 + x_3 + x_5 = 1\right\}$, $S_2 = \{1, 3, 4, 5, 6\}$.
(3) $\overline{Core}\left(S_3, v\right) = \left\{x \in \mathcal{X}(v) : x_2 + x_3 + x_6 = 1\right\}$, $S_3 = \{2, 3, 4, 5, 6\}$.

5.3. Controlled capacitated networks

Consider a scenario where individuals can collaborate to maximize the flow through an existing network characterized by links (such as pipes or channels) that have bounded capacities. In the maximal flow problem, a source node sends flow to a sink node, with each edge in the network having a defined capacity that limits the flow. The objective is to determine the maximum flow from the source to the sink while adhering to these capacity constraints. This concept has significant

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[4] An early reference for *mcst* games with revenue is chapter 5 in Moulin (2004), although it is not treated formally. An alternative interpretation (or an equivalent problem) arises when we consider that (instead of revenue) individuals have an *upper bound* on how much they are willing to pay to be connected.

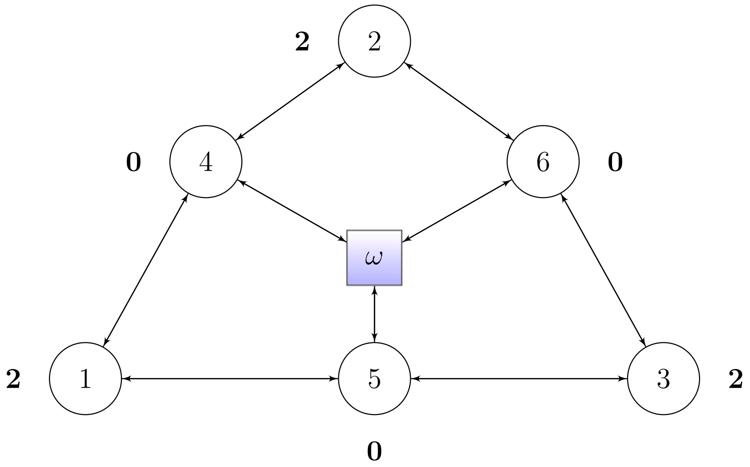

Fig. 2. *mcst* problem with revenue in Example 3. The cost of each link appearing in the picture is $c_{ij} = 1$, while the undrawn links have cost $c_{ij} = 2$. The revenue vector is $b = (2, 2, 2, 0, 0, 0)$. Finally, ω denotes the source.

applications in various real-world contexts, including electricity distribution and transportation networks, as well as in areas such as web communities, image segmentation, and telecommunications (Alsalami and Rushdi, 2021).

Controlled networks entail that in each link (*edge*) some coalition have the control to decide if such a link can be used or not. So, a *simple game* denoted by ω_e appears associated to any edge e in the network. These simple games controlling the edges are called *control games*. One can think of an almost literal example, where the edges represent oil pipelines, and the players are in power in different countries through which these pipelines cross (Peters, 2008). But whenever coalitions control the edges, the core of the cooperative game associated to this problem can be empty.

Example 4 applies the notion of δ-stable set in a problem of controlled capacitated network with an empty core. In these problems, the flow is maximal if all players cooperate, and then the question arises on how to distribute the profits. To ensure the non-emptiness of the core, all control games in a controlled capacitated network must have veto players (Peters, 2008).

Example 4 (*Peters, 2008*)**.** Let us consider the controlled capacitated network in Fig. 3. The source is denoted by h and the sink is denoted by t. There are four additional vertices denoted by ***a, b, c, d*** and any edge e joining two of these vertices have a capacity $k(e)$ that corresponds to the maximal amount that can flow through edge e.

There is a set of players $N = \{1, 2, 3\}$ that control the edges in the network in the following way:[5]

- $\omega_1(S) = 1$ if $1 \in S$ and $\omega_1(S) = 0$ otherwise
- $\omega_2(S) = 1$ if $2 \in S$ and $\omega_2(S) = 0$ otherwise
- $\omega_3(S) = 1$ if $3 \in S$ and $\omega_3(S) = 0$ otherwise
- $\omega_4(S) = \omega_5(S) = \omega_6(S) = 1$ if, and only if, $|S| = 2$
- $\omega_7(S) = 1$ if, and only if, $S \neq \emptyset$

The corresponding flow game assigns to any coalition $S \subseteq N$ a worth $v(S)$ equal to the maximal flow through the network containing only edges controlled by S. In this case, the flow game (N, v) coincides with the TU-game in Example 1, so the core is empty. The imputations in the δ-stable set are those in the black triangle of Fig. 1.

5.4. Sequencing games

An alternative example of cooperative game with possible an empty core comes from sequencing situations (Hamers et al., 1999; Slikker, 2006). Scheduling or sequencing problems originate from the processing and manufacturing industries, where multiple jobs must pass through various machines before production is finalized, such as in the production, assembly, and testing of consumer electronics. Beyond manufacturing, these problems also manifest in business, computing, and service sectors, exemplified by challenges like allocating airport landing slots for planes or scheduling computer programs (Hamers et al., 1999). In particular, we consider a scenario involving a multidivisional firm or several collaborating firms (the *agents*) utilizing a joint repair and maintenance facility. When service is needed, the agents incur costs due to downtime, and the aim is to minimize these costs by determining the optimal sequence in which the facility serves the agents. Additionally, the problem of allocating total downtime costs among the agents arises, leading to the formation of a cooperative game known as a *sequencing game*, where the core serves as a potential solution space for cost allocation. However, similar to previous scenarios, the core of this game may also be empty.

In particular, Hamers et al. (1999) and Slikker (2006) consider a multi-divisional firm in which the divisions share a repair and maintenance facility (or, equivalently, the case of several firms, for instance of car renting, that cooperate in a maintenance facility). Taking into consideration that the (financial) impact of a non-repaired item need not be the same for all cases, it might be beneficial to rearrange repair requests at the repair and maintenance facility rather than just using a first-come-first-serve principle (Slikker, 2006). The possible savings of this rearrangement should be shared among the divisions/firms (agents) that cooperate in the maintenance facility. The following example shows that the cooperative game associated with sequencing situations may have an empty core.

Example 5 (*Slikker, 2006*)**.** We focus on the allocation of the costs that five different divisions incur by analyzing cost allocation in sequencing situations with possible cost-savings by rearrangements in a setting with three parallel (and identical) machines. We denote the machines by capital letters A, B, C and the divisions (agents) by $N = \{1, 2, 3, 4, 5\}$. Each division is characterized by two parameters:

- The time t_i that needs the use of the machine for maintenance; $\tau = (t_1, t_2, t_3, t_4, t_5)$; and
- the cost of being in maintenance, that is assumed to be linear, $c_i(T) = \alpha_i T$, where T denotes the total time in maintenance; that is, for agent i, T_i equals to t_i plus the time this agent is waiting to enter the machine; let us denote $\Omega = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

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[5] For $r = 1, 2, \ldots, 7$, each ω_r is a simple game, that in this context is known as *control game*. The edge e_r can be used by coalition S if, and only if, coalition S controls that edge; that is, if $\omega_r(S) = 1$.

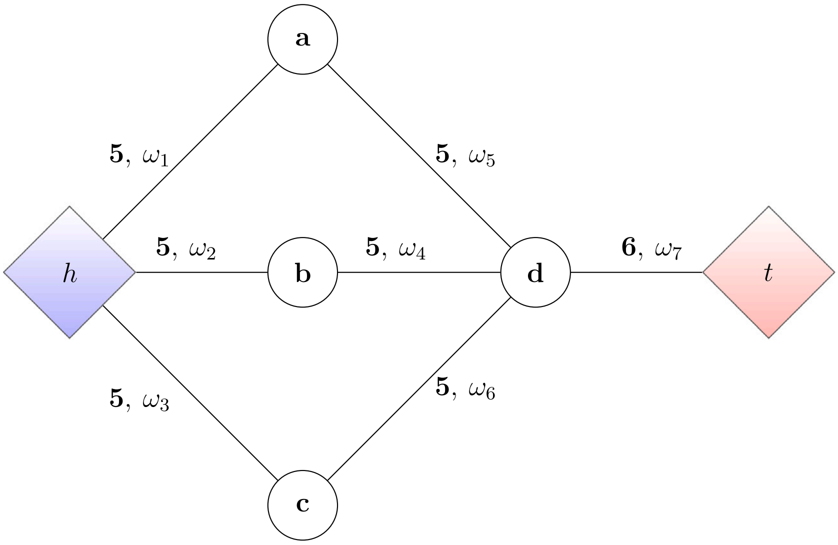

Fig. 3. Controlled capacitated network in Example 4.

In this example we have:

$\tau = (1, 3, 2, 1, 3) \qquad \Omega = (1, 3, 20, 1, 3)$

The initial order in which the jobs are processed (first-come-first-serve) and τ are: agent 1 → agent 3 → agent 4 → agent 2 → agent 5.

	A	B	C
Turn 1	Agent 1 (1 h)	Agent 3 (2 h)	Agent 4 (1 h)
Turn 2	Agent 2 (3 h)		Agent 5 (3 h)

Now consider coalition $S = \{1, 2\}$. The optimal order for this coalition is obtained by switching its two positions and then moving agent 1 to machine B before machine B completes its first task. The associated cost savings is $v(\{1, 2\}) = 1$. Applying symmetric reasoning to the coalition $\{4, 5\}$, (it is obvious that coalitions with only one agent have zero savings: $v(i) = 0$, for all i), the obtained TU-game is:

$$v(S) = \begin{cases} 1 & if \quad \{1,2\} \subseteq S, or\ \{4,5\} \subseteq S \\ 0 & otherwise \end{cases}$$

It is easy to check that the core of this TU-game is empty. The δ-stable set is

$$St^{\delta}(N, v) = \left\{ \left(x_1, x_2, x_3, x_4, x_5\right) \in \mathbb{R}^5_+ : \sum_{i=1}^{5} x_i = 1,\ x_3 = 0, \left(x_1 + x_2\right)\left(x_4 + x_5\right) = 0 \right\}$$

that is, agent 1 and agent 2 share the complete savings (and other agents do not receive anything), or agent 4 and agent 5 share the complete savings (and other agents do not receive anything) that seems a reasonable proposal in this problem.

6. Final comments

The results presented in this paper contribute to the broader understanding of cooperative solution concepts by proposing and analyzing the δ-stable set as an alternative in contexts where the core may be empty. We have proved that δ-stable allocations always exist, determine the core when it is not empty, and propose reasonable payoffs whenever the core is empty. Then, the set $St^{\delta}(N, v)$ could be considered as a pool containing the suitable solutions of general TU-games. By incorporating the dual perspective of objections and support, the δ-stable set captures a refined notion of collective acceptability. Its construction reflects the interplay between coalitional rationality and consistency, thereby offering a framework that preserves stability among supportive coalitions.

In order to define our dominance relation, which is the key of the solution, we suppose all agents have the same weight (that is why we just count how many players contain the coalition). The model could be adapted to players with different weight λ_i, $i \in N$ just by counting $p(S) = \sum_{i \in S} \lambda_i$, instead of $|S|$, in the definition of the HCR index $s(x)$.

CRediT authorship contribution statement

Begoña Subiza: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **José-Manuel Giménez-Gómez:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Josep E. Peris:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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