Results Math (2025) 80:130

1422-6383/25/050001-22
published online June 10, 2025
https://doi.org/10.1007/s00025-025-02450-1

Results in Mathematics

Mutual *d*-visibility in Graphs

Dorota Kuziak, Luis P. Montejano, and Juan A. Rodríguez-Velázquez

Abstract. Let d be a positive integer. The mutual d-visibility number $\mu^d(G)$ of a graph G is introduced as the cardinality of the largest mutual d-visibility set. That is, $X \subseteq V(G)$ is a mutual d-visibility set if for any pair of vertices $x, y \in X$, the distance between them is larger than d, or there exists a shortest x, y-path in G whose internal vertices are not in X. Several combinatorial and computational aspects of $\mu^d(G)$ are given in this work. Finally, the NP-completeness of the decision problem concerning finding $\mu^d(G)$ is proved.

Mathematics Subject Classification. 05C12, 05C69, 05C76.

Keywords. d-mutual-visibility number, d-mutual-visibility set, Mutual visibility, Dissociation number, Packing number, Product graphs.

1. Introduction

The mutual-visibility problem in networks is very well known in the area of computer science, in connection with mobile robots that navigate in a network avoiding collisions between them. It is understood that two robots are mutually visible if there is a shortest path between them in the network in which there is not located any other robot. There are several different approaches to the problem. Some of them considers the "visibility" properties between the navigating robots in such a way that any two robots must be mutually visible throughout all the possible shortest paths between them. This also means that among all the navigating robots, there are no three of them that are collinear. This idea becomes the so-called general position problem, which was independently introduced in [15, 22], although some first works considering it as a pure combinatorial problem in hypercubes was already known from [17].

In addition, it is understood that the origin of the general position problem is the celebrated still open "no-three-in-line" problem posed by Henry Dudeney in [8].

Another approach to the visibility problem requires the mutual visibility properties to be satisfied only through at least one shortest path between the navigating robots. This approach was first considered from a graph theory point of view in [7], but it has some origin in previous computer science problems in which some robot navigation models avoiding collisions were presented. Among such works we find for instance [1,6,9,20]. Notice that somehow, also the "no-three-in-line" problem can be taken as an origin of the this mutual-visibility variation.

In formal way, given a connected graph G and a set of vertices $X \subseteq V(G)$, it is said that two vertices $x, y \in V(G)$ are X-*visible*, if there is a shortest x, y-path in G whose internal vertices are not in X. Now, the set X is called a *mutual-visibility set* of G if all the vertices of X are pairwise X-visible. The *mutual-visibility number* of G is the cardinality of a largest possible mutual-visibility set of G, and is denoted by $\mu(G)$. A mutual-visibility set of cardinality $\mu(G)$ is usually represented in the literature as a μ-*set*.

The investigation on the mutual-visibility problem has recently continued from a theoretical point of view, and one can find now a few interesting investigations considering several combinatorial properties of the problem. Among them we have for instance [2–5,18,21]. We remark the fact that along the study of the mutual-visibility problems, it has been required the definition of some few variations of the standard mutual-visibility concept. For example, in [3], the authors considered the mutual-visibility of Cartesian product graphs, and to this end, the concept of total mutual-visibility number was needed.

We say that a set of vertices X is *total mutual-visibility set* if any two vertices of the graph are X-visible. The total mutual-visibility number is then defined in a natural way, as its antecessor, and is denoted by $\mu_t(G)$. That is, in this new version the "visibility" of a set S of vertices in a graph was extended to be satisfied not only between the vertices of S, but also between all the vertices of the graph, with respect to the set S.

We next consider a different point of view for the mutual-visibility problem. This comes as follows. Networks can be sometimes very large, and the navigation of robots in such networks can have some different limitations, like for instance, lack of range to reach all the points in the network. This would mean that a robot can have somehow a limited "visibility" and other robots that are at some enough far distance cannot be taken into account for the "visibility properties". This suggest the idea of considering the visibility between the navigating robots with some kind of locality, or equivalently, considering the visibility only between robots that are at a bounded distance.

According to this, for a given connected graph G, a set $X \subseteq V(G)$, and an integer $d \geq 1$, two vertices $x, y \in X$ are (X, d)-*visible* in G, if the distance between these vertices is larger than d, or there exists a shortest x, y-path in

G whose internal vertices are not in X. Also, the set of vertices X is a *mutual d-visibility set* if any two vertices in X are (X, d)-*visible*. A largest mutual d-visibility set of G is a $\mu^d(G)$-*set*, and its cardinality is the *mutual d-visibility number* of G, denoted by $\mu^d(G)$.

Next we present some additional notation and terminology, which is necessary to develop our exposition. We study here only connected and undirected graphs. Given a graph G, the notation $\mathrm{d}_G(x, y)$ (or $\mathrm{d}(x, y)$ if it is clear from the context) represents the *distance* between $x, y \in V(G)$, that is the length of a shortest x, y-path. The largest distance between any pair of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. The set $N_G(v)$ ($N_G[v]$) is the *open neighborhood* (*closed neighborhood*) of a vertex $v \in V(G)$. Similarly, for $C \subseteq V(G)$, the set of vertices in $V(G) \setminus C$ adjacent to some vertex of C is denoted by $N(C)$. The *order* of G is written as $\mathrm{n}(G)$. The *independence number* of a graph G is denoted by $\alpha(G)$, and it represents the cardinality of a largest set of vertices which induces an empty graph. An $\alpha(G)$-*set* is an independent set of cardinality $\alpha(G)$. Moreover, given two graphs G and H, the *join graph* $G + H$ is a graph formed from the disjoint union of G and H by adding an edge between each vertex of G and each vertex of H. For the sake of readability, notation and terminology that is only used in specific places in the paper will be introduced just before it is used.

The plan of our article is as follows. In Section 2, we describe some relationships between $\mu^d(G)$ and other known concepts and parameters. As examples, we determine the exact value of $\mu^d(G)$ when G is a path or a cycle. Section 3 is devoted to obtain lower and upper bounds for $\mu^d(G)$ in terms of parameters like the k-dissociation number and the k-packing number, among others. In addition, in Section 3, we present upper bounds for $\mu^d(G)$ involving convex subgraphs of G and, as a consequence of this, we obtain a closed formula for the case of corona product graphs in Section 5. The subsequent section is dedicated to characterize the limit case of $\mu^d(G)$ with respect to the order and the number of simplicial vertices of G. The mutual d-visibility number of the corona and lexicographic products of two graphs G and H is studied in Section 5, where we give the exact value of this parameter in such products. In the penultimate section, we prove the NP-completeness of the decision problem associated to the optimization problem of determining the mutual d-visibility number of arbitrary graphs. Finally, in the Concluding remarks section, we propose a list of open problems and possible research lines.

2. First Basic Remarks

It is clear that if $d = 1$, then the vertices of the whole graph is the $\mu^1(G)$-set, and if $d \geq \mathrm{diam}(G)$, then any $\mu^d(G)$-set represents a standard mutual-visibility set as defined in [7]. One interesting case is that of graphs G with diameter two, in which the values of d to be considered are only $d = 1$ and $d = 2$, as diam$(G) = 2$ leads to $\mu^2(G) = \mu(G)$. For instance, among these graphs we have the case of join graphs, complete bipartite graphs and Hamming graphs, whose mutual-visibility numbers were studied in [3,7].

Related to Hamming graphs, we may remark the following situation. It was proved in [3], that the problem of computing the mutual visibility number of the Cartesian product of two complete graphs K_m and K_n is equivalent to the problem of solving one instance of the well known Zarankiewicz's problem, which is stated as follows. If m, n, s, t are positive integers, then we need to find the maximum number $z(m, n; s, t)$ of 1's that an $m \times n$ binary matrix can have, provided that it contains no constant $s \times t$ submatrix of 1's. Based on the comments above, and the results of [3], we also have that, for any integers $m, n \geq 2$,

$$\mu^2(K_m \square K_n) = z(m, n; 2, 2).$$

By all the reasons mentioned above, from now on we focus our attention on the cases $2 \leq d \leq \text{diam}(G) - 1$, and on graphs whose diameter is larger than two. Moreover, the following chain of inequalities clearly holds.

$$\mu(G) = \mu^{\text{diam}(G)}(G) \leq \mu^{\text{diam}(G)-1}(G) \leq \cdots \leq \mu^2(G) \leq \mu^1(G) = \text{n}(G) \quad (1)$$

The topic of mutual d-visibility in graphs introduced in this paper is related to the general d-position problem in graphs, which was studied in [16]. The difference between them is that the mutual d-visibility properties need to be satisfied only through at least one shortest path, in contrast to the general d-position properties which are satisfied by all shortest paths. Let G be a connected graph and let $d \geq 1$ be an integer. A set $X \subseteq V(G)$ is a *general d-position set* of G if no three different vertices in X lie on a common shortest path of length at most d. The cardinality of a largest general d-position set in G is the *general d-position number* of G, denoted by $\text{gp}_d(G)$. Clearly, any general d-position set is also a mutual d-visibility set for any connected graph G and any integer $d \geq 1$. In this sense, the following bound is directly obtained.

$$\mu^d(G) \geq \text{gp}_d(G) \quad (2)$$

We next recall two known results on $\text{gp}_d(G)$, which are needed in our exposition.

Proposition 2.1 [16]. *If $n \geq 3$ and $2 \leq d \leq n-1$, then*

$$\text{gp}_d(P_n) = \begin{cases} 2\left\lceil \frac{n}{d+1} \right\rceil - 1 & n \equiv 1 \bmod (d+1), \\ 2\left\lceil \frac{n}{d+1} \right\rceil & \textit{otherwise.} \end{cases}$$

Proposition 2.2 [16]. *If $n \geq 5$ and $2 \leq d \leq \lfloor n/2 \rfloor$, then*

$$\text{gp}_d(C_n) = \begin{cases} 2\left\lfloor \frac{n}{d+1} \right\rfloor + 1 & n \equiv d \bmod (d+1), \\ 2\left\lfloor \frac{n}{d+1} \right\rfloor & \textit{otherwise.} \end{cases}$$

If $d \geq \lfloor n/2 \rfloor$, *then* $\mathrm{gp}_d(C_n) = 3$.

Notice that odd cycles and trees (and paths among them) are graphs with exactly one shortest path between any two vertices. In this case, X is a mutual d-visibility set if and only if X is a general d-position set. As direct consequences of this fact, the following two results are respectively obtained from Propositions 2.1 and 2.2.

Corollary 2.3. *If* $n \geq 3$ *and* $2 \leq d \leq n-1$, *then*

$$\mu^d(P_n) = \begin{cases} 2\left\lceil \frac{n}{d+1} \right\rceil - 1 & n \equiv 1 \bmod (d+1), \\ 2\left\lceil \frac{n}{d+1} \right\rceil & \textit{otherwise.} \end{cases}$$

In order to give the value of $\mu^d(C_n)$, notice that only the part related to odd cycles is directly obtained from Proposition 2.2. For the even cycles, we only need to consider one extra situation when $d \geq n/2 = \mathrm{diam}(C_n)$ because for any two diametral vertices there is not a unique shortest between them. However, in such case it can be readily seen that $\mu^d(C_n) = 3$. If $2 \leq d < \mathrm{diam}(C_n)$, then there is only one shortest path between any two vertices at distance at most d, and the arguments above again apply.

Corollary 2.4. *If* $n \geq 5$ *and* $2 \leq d \leq \lfloor n/2 \rfloor$, *then*

$$\mu^d(C_n) = \begin{cases} 2\left\lfloor \frac{n}{d+1} \right\rfloor + 1 & n \equiv d \bmod (d+1), \\ 2\left\lfloor \frac{n}{d+1} \right\rfloor & \textit{otherwise.} \end{cases}$$

If $d \geq \lfloor n/2 \rfloor$, *then* $\mu^d(C_n) = 3$.

3. General Bounds

In this section, some general bounds for the mutual d-visibility number in graphs are presented. The results are related to other parameters which are defined below.

Given a positive integer k and a graph G, a non-empty set $X \subseteq V(G)$ is a k-*dissociation set* of G if the following two conditions hold for every pair of vertices $x, y \in X$.

- $\mathrm{d}(x,y) = 1$ or $\mathrm{d}(x,y) > k$,
- $X \cap N(x) \cap N(y) = \varnothing$.

A largest k-dissociation set of G is called a $\mathrm{diss}_k(G)$-*set*, and its cardinality is the k-*dissociation number* of G, denoted by $\mathrm{diss}_k(G)$. This concept was introduced by Yannakakis in [23] for the particular case of $k = 1$, where the parameter $\mathrm{diss}_1(G)$ was called the *dissociation number* of G.

A subset $X \subseteq V(G)$ is a k-*packing* of G if $\mathrm{d}(x,y) > k$ for all pairs of distinct vertices $x, y \in X$ [19]. A largest k-packing of G is called a $\rho_k(G)$-*set* and its cardinality is denoted by $\rho_k(G)$.

A vertex of a graph G is *simplicial* if its neighbourhood induces a complete subgraph of G. The set of simplicial vertices of G will be denoted by $\mathcal{S}(G)$.

With these concepts and notation in mind, we can develop this section. In particular, the following result involves the three concepts described above.

Theorem 3.1. *For any connected graph G of order $\mathrm{n}(G) \geq 3$ and any integer $d \geq 2$,*

$$\mu^d(G) \geq \max\{\mathrm{diss}_{d-1}(G), \rho_{\lfloor d/2 \rfloor}(G), |\mathcal{S}(G)|\}.$$

Proof. Let X be a $\mathrm{diss}_{d-1}(G)$-set. We will see that X is a mutual d-visibility set of G. For any pair of vertices $u, v \in X$ we have that $\mathrm{d}(u,v) \geq d$ or $\mathrm{d}(u,v) = 1$. If $\mathrm{d}(u,v) > d$ or $\mathrm{d}(u,v) = 1$, then clearly u and v are (X,d)-visible. Now suppose that $\mathrm{d}(u,v) = d$ and that u and v are not (X,d)-visible. Then, for every shortest $\{u,v\}$-path P, there exists $w \in X \cap V(P)$. Hence, $\mathrm{d}(u,w) = 1$ and $\mathrm{d}(w,v) = d-1$ or $\mathrm{d}(u,w) = d-1$ and $\mathrm{d}(w,v) = 1$, concluding that $d = 2$. Thus, $N(u) \cap N(v) \cap X \neq \varnothing$, yielding a contradiction. Therefore, X is a mutual d-visibility set of G and the bound $\mu^d(G) \geq |X| = \mathrm{diss}_{d-1}(G)$ follows.

On the other hand, since no simplicial vertex is an internal vertex of a shortest path, either $\mathcal{S}(G) = \varnothing$ or $\mathcal{S}(G)$ is a mutual d-visibility set of G, and so $\mu^d(G) \geq |\mathcal{S}(G)|$.

In order to prove $\mu^d(G) \geq \rho_{\lfloor d/2 \rfloor}(G)$, we consider a $\rho_{\lfloor d/2 \rfloor}(G)$-set Y. If Y is not a mutual d-visibility set of G, then there exist two vertices $u, v \in Y$ which are not (Y,d)-visible. Hence, for every shortest path $u = u_0, u_1, \ldots, u_r = v$, there exits $w \in Y \cap \{u_1, \ldots, u_{r-1}\}$ $(r \leq d)$. Notice that this is a contradiction as in such a case $\mathrm{d}(u,w) \leq \lfloor d/2 \rfloor$ or $\mathrm{d}(w,v) \leq \lfloor d/2 \rfloor$, which is impossible as Y is a $\rho_{\lfloor d/2 \rfloor}(G)$-set. Therefore, $\mu^d(G) \geq |Y| = \rho_{\lfloor d/2 \rfloor}(G)$. □

The bounds above are tight. For instance, it is not difficult to check that for any path $\mu^d(P_n) = \mathrm{diss}_{d-1}(P_n)$, while for any star graph $\mu^2(K_{1,n}) = \mathrm{diss}_1(K_{1,n}) = |\mathcal{S}(K_{1,n})| = \rho_1(K_{1,n}) = n$ for every $n \geq 2$. A characterization of graphs with $\mu^d(G) = |\mathcal{S}(G)|$ for $d \in \{2,3\}$ will be given in Section 4.

Since the k-dissociation number has not been previously studied for any $k \geq 2$, we proceed to derive a lower bound, which immediately provides a lower bound for $\mu^d(G)$. In the next result, $L(G)$ will denote the line graph of G, i.e., the graph whose vertex set is the edge set of G, where two vertices of $L(G)$ are adjacent if the corresponding edges in G have a vertex in common.

Theorem 3.2. *For any connected graph G and any integer $d \geq 2$,*

$$\mathrm{diss}_{d-1}(G) \geq 2\rho_d(L(G)).$$

Proof. Let X be a $\rho_d(L(G))$-set and let $Y = \{x \in V(G) : \ x \in e \text{ for some } \ e \in X\}$. Since $X \subseteq E(G)$ and $d_{L(G)}(e,e') > d$, for every $e = xy, e' = x'y' \in E(G) \cap X$ we have that

$$\begin{aligned} d_G(\{x,y\},\{x',y'\}) &= \min\{d_G(x,x'), d_G(x,y'), d_G(y,x'), d_G(y,y')\} \\ &= d_{L(G)}(e,e') - 1 > d - 1. \end{aligned}$$

Therefore, Y is a $(d-1)$-dissociation set of G, which implies that $\mathrm{diss}_{d-1}(G) \geq |Y| = 2|X| = 2\rho_d(L(G))$. □

In order to show a family of graphs where $\mu^d(G) = \mathrm{diss}_{d-1}(G) = 2\rho_d(L(G))$, we observe that for any path P_n of order $n \not\equiv 1 \bmod (d+1)$ and $2 \leq d \leq n-1$ we have

$$2\lceil \frac{n}{d+1} \rceil = \mu^d(P_n) = \mathrm{diss}_{d-1}(P_{n-1}) = 2\rho_d(P_{n-1}) = 2\lceil \frac{n-1}{d+1} \rceil.$$

The set $\mathcal{S}(G)$ of simplicial vertices of a graph G can be partitioned into true twin equivalence classes. Two vertices $u, v \in \mathcal{S}(G)$ belong to the same true twin equivalence class if and only if they are true twins, i.e., whenever $N_G[u] = N_G[v]$. We distinguish two sets of true twin equivalence classes, the set $\mathcal{CS}_1(G)$ of singleton classes and the set $\mathcal{CS}_2(G)$ of non-singleton classes.

Lemma 3.3. *For any connected graph G, there exists a $\mathrm{diss}_1(G)$-set X satisfying the following conditions.*

(a) *$u \in X$ for every singleton class $\{u\} \in \mathcal{CS}_1(G)$,*
(b) *$|C \cap X| = 2$ for every non-singleton class $C \in \mathcal{CS}_2(G)$.*

Proof. Let X' be any $\mathrm{diss}_1(G)$-set. If there exists a singleton class $\{u\} \in \mathcal{CS}_1(G)$ such that $u \notin X'$, then $N(u) \cap X' \neq \varnothing$. Thus, for any $v \in N(u) \cap X'$, we have that $(X' \cup \{u\}) \setminus \{v\}$ is a $\mathrm{diss}_1(G)$-set. Now, suppose that there exists a non-singleton class $C' \in \mathcal{CS}_2(G)$ such that $C' \cap X' = \varnothing$ and observe that $|N(C') \cap X'| \leq 2$, otherwise X' would not be a 1-dissociation set. If $|N(C') \cap X'| \leq 1$, then we have a contradiction with the definition of $\mathrm{diss}_1(G)$-set. Now, if $|N(C') \cap X'| = 2$, then for the two vertices x, x' belonging to $N(C') \cap X'$, and any two vertices y, y' belonging to C', we have that $(X' \cup \{y, y'\}) \setminus \{x, x'\}$ is a $\mathrm{diss}_1(G)$-set. Finally, if there exists a class $C'' \in \mathcal{CS}_2(G)$ such that $C'' \cap X' = \{w\}$, then there exists $z \in (N(w) \cap X') \setminus C''$. Hence, for any $t \in C'' \setminus \{w\}$, we have that $(X' \cup \{t\}) \setminus \{z\}$ is a $\mathrm{diss}_1(G)$-set. Therefore, the construction of X from X' can be done by following the process described for the previous cases. In addition, we may also recall that $|C \cap X| > 2$ is not possible because this would violate the definition of X being a $\mathrm{diss}_1(G)$-set. □

Proposition 3.4. *For any connected graph G with $\mathrm{n}(G) \geq 3$,*

$$\mu^2(G) \geq \mathrm{diss}_1(G) + |\mathcal{S}(G)| - 2|\mathcal{CS}_2(G)| - |\mathcal{CS}_1(G)|.$$

Proof. Let X be a $\mathrm{diss}_1(G)$-set satisfying Lemma 3.3. Since $X \cup \mathcal{S}(G)$ is a mutual 2-visibility set of G, by the inclusion-exclusion principle,

$$\mu^2(G) \geq |X \cup \mathcal{S}(G)| = \mathrm{diss}_1(G) + |\mathcal{S}(G)| - 2|\mathcal{CS}_2(G)| - |\mathcal{CS}_1(G)|.$$

□

The bound above is tight. For instance, consider the graph $G_{k,r,s}$ obtained from a path P_{3k+1} and two complete graphs K_r and K_s, where we identify one leaf of P_{3k+1} with one vertex of K_r and the other leaf with one vertex of K_s. For $k \geq 1$, $r \geq 3$ and $s \geq 3$, we have $\mu^2(G_{k,r,s}) = 2k+r+s-2$, $\mathrm{diss}_1(G_{k,r,s}) = 2k+4$, $|\mathcal{S}(G_{k,r,s})| = r+s-2$, $|\mathcal{CS}_2(G_{k,r,s})| = 2$ and $|\mathcal{CS}_1(G_{k,r,s})| = 0$. Hence,

$$\mu^2(G_{k,r,s}) = \mathrm{diss}_1(G_{k,r,s}) + |\mathcal{S}(G_{k,r,s})| - 2|\mathcal{CS}_2(G_{k,r,s})| - |\mathcal{CS}_1(G_{k,r,s})|.$$

The bound is also achieved for trees, which will be stated in Corollary 3.6. We will also show in Section 4 that the bound above is reached for the family of graphs satisfying Proposition 4.4 (ii).

Until now, we have provided some lower bounds for the mutual d-visibility number of graphs. We next concentrate on finding upper bounds for such parameter.

Proposition 3.5. *For any connected graph G of girth $g(G) \geq 5$ and any integer $d \geq 2$,*

$$\mu^d(G) \leq \mathrm{diss}_1(G) \text{ and } \mu^2(G) = \mathrm{diss}_1(G).$$

Proof. Let X be a $\mu^d(G)$-set. Suppose there exist $x, y, z \in X$ such that $y, z \in N(x)$. In this case either $y \sim z$ or there exists $w \in V(G) \setminus X$ such that $w \in N(y) \cap N(z)$, and so $g(G) \leq 4$. Therefore, $g(G) \geq 5$ implies that X is a 1-dissociation set of G and, as a result, $\mu^d(G) = |X| \leq \mathrm{diss}_1(G)$. Moreover, if $d = 2$, then by Theorem 3.1 we have that $\mu^2(G) = \mathrm{diss}_1(G)$. □

First notice that $\mu^2(C_5) = 3 = \mathrm{diss}_1(C_5)$. Moreover, consider a graph F' obtained from two disjoint cycles C_5 by adding one extra edge between any two vertices from each these cycles (sometimes also called Kayak-Paddle graphs in the literature). Hence, $\mu^2(F') = \mu^3(F') = 6 = \mathrm{diss}_1(F')$, and the bound from Proposition 3.5 is achieved.

Since a tree has no cycles, it is assumed the agreement that $g(T) = +\infty$ for every tree T. Hence, Proposition 3.5 immediately gives the following consequence.

Corollary 3.6. *For any tree T and any integer $d \geq 2$,*

$$\mu^d(T) \leq \mathrm{diss}_1(T) \text{ and } \mu^2(T) = \mathrm{diss}_1(T).$$

We next give upper bounds for $\mu^d(G)$ involving convex subgraphs of G. A *convex subgraph* of a graph G is a subgraph H of G that includes every shortest path in G between any pair of vertices of H.

Theorem 3.7. *Given a connected graph G and any integer $d \geq 2$, the following statements hold.*

(i) *If H is a* convex subgraph *of G, then*

$$\mu^d(G) \leq \mathrm{n}(G) - \mathrm{n}(H) + \mu^d(H).$$

(ii) *If there exist a family* $V_1, \ldots, V_k$ *of subsets of* $V(G)$*, whose respective induced subgraphs* $H_1, \ldots, H_k$ *are convex, and* $V(G) = \bigcup_{i=1}^k V_i$*, then*

$$\mu^d(G) \leq \sum_{i=1}^{k} \mu^d(H_i).$$

Proof. Let X be a $\mu^d(G)$-set. If H is a convex subgraph of G, then $X \cap V(H)$ is a mutual d-visibility set of H, which implies that $\mu^d(H) \geq |X \cap V(H)|$. Therefore, $\mu^d(G) = |X| \leq \mathrm{n}(G) - \mathrm{n}(H) + \mu^d(H)$.

Analogously, if the subgraphs $H_1, \ldots, H_k$, induced by $V_1, \ldots, V_k$ are convex, then $X \cap V_i$ is a mutual d-visibility set of H_i for every $i \in \{1, \ldots, k\}$. Therefore, if $V(G) = \cup_{i=1}^k V_i$, then

$$\mu^d(G) = |X| \leq \sum_{i=1}^{k} |X \cap V_i| \leq \sum_{i=1}^{k} \mu^d(H_i).$$

□

The bounds above are tight. For instance, consider the graph $G_{r,s}$ obtained from a path P_{d+1} and two complete graphs K_r and K_s, where we identify one leaf of P_{d+1} with one vertex of K_r and the other leaf with one vertex of K_s. For $r \geq 2$ and $s \geq 2$ we take a convex subgraph H of $G_{r,s}$, isomorphic to a diametral path P_{d+3}, and so we have that $\mu^d(G_{r,s}) = r + s$, $\mu^d(H) = 4$, $\mathrm{n}(G_{r,s}) = d + r + s - 1$ and $\mathrm{n}(H) = d + 3$. Hence, $\mu^2(G_{r,s}) = \mathrm{n}(G_{r,s}) - \mathrm{n}(H) + \mu^d(H)$, which shows that the bound given by Theorem 3.7 (i) is tight. An example showing the tightness of the Theorem 3.7 (ii) shall be given in Proposition 5.1. Observe that for graphs having a diametral path whose vertices induce a convex subgraph, the bound given by Theorem 3.7 (i) coincides with the bound that we next prove for arbitrary graphs of diameter k.

Theorem 3.8. *For any connected graph* G *of diameter* k *and any integer* $d \geq 2$*,*

$$\mu^d(G) \leq \mathrm{n}(G) - k - 1 + \mu^d(P_{k+1}).$$

Proof. Let $u, v \in V(G)$ be two diametral vertices and X a $\mu^d(G)$-set. Suppose that X contains more than $\mu^d(P_{k+1})$ vertices from each shortest u, v-paths. Let P_{k+1} be a shortest u, v-path such that $r = |X \cap V(P_{k+1})|$ is minimum among all shortest u, v-paths. Let $V(P_{k+1}) = \{u_0, u_1 \ldots, u_k\}$, where consecutive vertices are adjacent. Since $r > \mu^d(P_{k+1})$, there exist $u_i, u_j, u_l \in X \cap V(P_{k+1})$, where $i < j < l$ and $\mathrm{d}(u_i, u_l) \leq d$. Hence, there exists a shortest u_i, u_l-path $u_i = v_0, \ldots, v_{l-i} = u_l$ such that $X \cap \{v_1, \ldots, v_{l-i-1}\} = \varnothing$. Thus, $u = u_0, \ldots, u_i, v_1, \ldots, v_{l-i-1}, u_l, \ldots, u_k = v$ is a shortest u, v-path P'_{k+1}, where $|X \cap V(P'_{k+1})| < r$, which is a contradiction. Hence, there exists a diametral path $u_0, u_1 \ldots, u_k$ such that $|X \cap \{u_0, u_1, \ldots, u_k\}| = r \leq \mu^d(P_{k+1})$. Therefore, $\mu^d(G) \leq \mathrm{n}(G) - (k + 1 - r) \leq \mathrm{n}(G) - k - 1 + \mu^d(P_{k+1})$. □

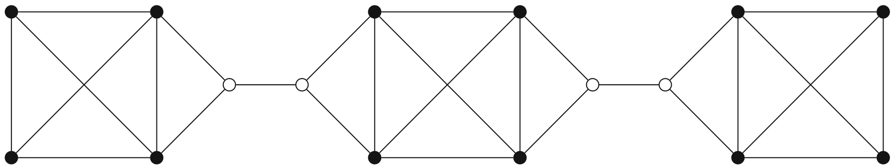

Figure 1. A graph G_0 of diameter $k = 9$ with $\mu^3(G_0) = \mathrm{n}(G_0) - k - 1 + \mu^3(P_{k+1})$

The bound above is tight. For instance, let G_r be the graph obtained from the graph G_0 shown in Figure 1 by subdividing $r \geq 0$ times the edges formed by white coloured vertices. In this case, $n(G_r) = 16 + 2r$, $k = 9 + 2r$ and for any $d = r + 3$ we obtain $\mu^d(G_r) = 12$ and $\mu^d(P_{10+2r}) = 6$. Hence, $\mu^d(G_r) = \mathrm{n}(G_r) - k - 1 + \mu^d(P_{k+1})$.

4. Some Extreme Cases

First, we characterize the limit cases of $\mu^d(G)$ with respect to the order of G. That is, we know that $\mu^d(G) \leq \mathrm{n}(G)$, and we are interested in the cases $\mu^d(G) = \mathrm{n}(G)$ and $\mu^d(G) = \mathrm{n}(G) - 1$. Although a graph without vertices is not defined, for the sake of brevity, in the statement of the next result we will use the notation $G \cong (K_1 \cup K_r) + G'$ for any integer $r \geq 0$, where we will assume the convention $K_1 \cup K_0 = K_1$.

Proposition 4.1. *Given a connected graph G, the following statements hold.*

(i) $\mu^d(G) = \mathrm{n}(G)$ *if and only if G is a complete graph or $d = 1$.*

(ii) $\mu^d(G) = \mathrm{n}(G) - 1$ *if and only if $d \geq 2$ and $G \cong (K_1 \cup K_r) + G'$, where G' is a non-complete graph and r is a non-negative integer.*

Proof. It is clear that if $d = 1$ or G is a complete graph, then $\mu^d(G) = \mathrm{n}(G)$. From now on we assume that G is a non-complete graph and $d \geq 2$. Thus, Theorem 3.8 leads to

$$\mu^d(G) \leq \mathrm{n}(G) - 1. \tag{3}$$

Thus, from (3) we conclude that (i) follows.

Now, if $G \cong (K_1 \cup K_r) + G'$ for some non-complete graph G' and some non-negative integer r, then we have that $V(G) \setminus V(K_1)$ is a mutual d-visible set of G, and by (3) we conclude that $\mu^d(G) = \mathrm{n}(G) - 1$. Conversely, if $\mu^d(G) = \mathrm{n}(G) - 1$ then there exists a vertex $v \in V(G)$ such that $V(G) \setminus \{v\}$ is a $\mu^d(G)$-set, which implies that x, v, y is a shortest path for every pair of non-adjacent vertices $x, y \in V(G) \setminus \{v\}$. Therefore, $G = (K_1 \cup K_r) + G'$, where $V(K_1) = \{v\}$, $V(K_r) = V(G) \setminus N_G[v]$ and $V(G') = N_G(v)$. Therefore, (ii) follows. □

Now we are interested in the case of non-complete graphs G with $\mu^d(G) = |\mathcal{S}(G)|$.

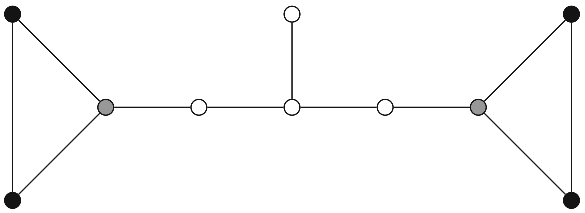

FIGURE 2. A graph with $\mu^4(G) = 6 > |\mathcal{S}(G)| = 5$. Here every non-simplicial vertex lies on the unique shortest path between simplicial vertices at distance 4. The grey vertices are the non-simplicial vertices in the mutual 4-visibility set

Proposition 4.2. *If G is a connected graph with $\mu^d(G) = |\mathcal{S}(G)|$ for $d \geq 2$, then for every vertex $w \in V(G) \setminus \mathcal{S}(G)$ there exists a pair of vertices $u, v \in \mathcal{S}(G)$ with $\mathrm{d}(u, v) \leq d$, such that w is an internal vertex of every shortest u, v-path.*

Proof. If there exists a vertex $w \in V(G) \setminus \mathcal{S}(G)$ such that for every pair of vertices $u, v \in \mathcal{S}(G)$ with $\mathrm{d}(u, v) \leq d$, there exists a shortest u, v-path not containing w, then $\mathcal{S}(G) \cup \{w\}$ is a mutual d-visibility set of G and so, $\mu^d(G) \geq |\mathcal{S}(G)| + 1$. Therefore, the result follows. □

As the following figure shows, the converse of Proposition 4.2 does not hold.

Lemma 4.3. *For any connected graph G, there exists a $\mu^2(G)$-set X such that $\mathcal{S}(G) \subseteq X$.*

Proof. Let X be a $\mu^2(G)$-set such that $|X \cap \mathcal{S}(G)|$ is maximum among all $\mu^2(G)$-sets. Suppose that there exists $w \in \mathcal{S}(G) \setminus X$. If $N_G(w) \cap (X \setminus \mathcal{S}(G)) = \varnothing$, then $X' = X \cup \{w\}$ is a mutual 2-visibility set with $|X'| > |X| = \mu^2(G)$, which is a contradiction. Now, if there exists a vertex $u \in N_G(w) \cap (X \setminus \mathcal{S}(G))$, then the set $X'' = (X \cup \{w\}) \setminus \{u\}$ is a $\mu^d(G)$-set with $|X'' \cap \mathcal{S}(G)| > |X \cap \mathcal{S}(G)|$, which is a contradiction. Therefore, the result follows. □

Next we characterize the non-complete graphs G having $\mu^2(G) = |\mathcal{S}(G)|$.

Proposition 4.4. *Given a connected non-complete graph G, the following statements are equivalent.*

(i) $\mu^2(G) = |\mathcal{S}(G)|$.
(ii) *For every vertex $w \in V(G) \setminus \mathcal{S}(G)$, there exist two vertices $u, v \in \mathcal{S}(G)$, belonging to distinct classes of simplicial vertices, such that $N_G(u) \cap N_G(v) = \{w\}$.*

Proof. By Lemma 4.3 there exists a $\mu^2(G)$-set X such that $\mathcal{S}(G) \subseteq X$. Hence, if (ii) holds, then $X \setminus \mathcal{S}(G) = \varnothing$. Therefore, $\mu^2(G) = |\mathcal{S}(G)|$.

Conversely, if $\mu^2(G) = |\mathcal{S}(G)|$, then by Proposition 4.2 we conclude that (ii) follows. □

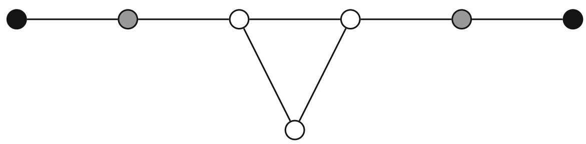

Figure 3. A graph with $\mu^3(G) = 4 > 3 = |\mathcal{S}(G)|$. In this case, $\mathcal{S}(G) \setminus X \neq \varnothing$ for every $\mu^3(G)$-set X

Notice that for the graphs G satisfying Proposition 4.4 (ii) it follows that $\mathrm{diss}_1(G) = 2|\mathcal{CS}_2(G)| + |\mathcal{CS}_1(G)|$. Hence, the bound given in Proposition 3.4 is achieved by graphs with $\mu^2(G) = |\mathcal{S}(G)|$.

As the next figure shows, for $d = 3$ does not exist a similar result to Lemma 4.3.

Proposition 4.5. *Let G be a non-complete graph. If for every vertex $w \in V(G) \setminus \mathcal{S}(G)$, there exists a vertex $v \in \mathcal{S}(G)$ such that $N_G(v) \setminus \mathcal{S}(G) = \{w\}$, then $\mu^d(G) = |\mathcal{S}(G)|$ for every $d \geq 3$.*

Proof. Let G be a graph satisfying the assumptions, and let X be a $\mu^d(G)$-set such that $|X \cap \mathcal{S}(G)|$ is maximum among all $\mu^d(G)$-sets with $d \geq 3$. Suppose that there exists $w \in X \setminus \mathcal{S}(G)$. Let $W = \{w' \in \mathcal{S}(G) : \ N_G(w') \setminus \mathcal{S}(G) = \{w\}\}$.

Notice that $W \subseteq X$, by the maximality of $|X \cap \mathcal{S}(G)|$. Obviously, W forms a class true twins in G and, since G is not a complete graph, there exists $u \in N_G(w) \setminus \mathcal{S}(G)$. Let $u' \in \mathcal{S}(G)$ such that $N_G(u') \setminus \mathcal{S}(G) = \{u\}$. Since $d \geq 3$ and u', u, w, w' is the only shortest path from u' to any vertex $w' \in W$, we have that $u' \notin X$. Hence, $X' = (X \cup \{u'\}) \setminus \{w\}$ is a $\mu^d(G)$-set with $|X' \cap \mathcal{S}(G)| > |X \cap \mathcal{S}(G)|$, which is a contradiction. Therefore, $X = \mathcal{S}(G)$. □

Proposition 4.6. *If G is a non-complete graph, then the following statements are equivalent.*

(i) $\mu^3(G) = |\mathcal{S}(G)|$.
(ii) *For every vertex $w \in V(G) \setminus \mathcal{S}(G)$, there exists a vertex $v \in \mathcal{S}(G)$ such that $N_G(v) \setminus \mathcal{S}(G) = \{w\}$.*

Proof. First, we assume that $\mu^3(G) = |\mathcal{S}(G)|$. Obviously, $\mathcal{S}(G)$ is a $\mu^3(G)$-set. Hence, if there exists a vertex $w \in V(G) \setminus \mathcal{S}(G)$, such that $N_G(v) \setminus \mathcal{S}(G) \neq \{w\}$ for every vertex $v \in \mathcal{S}(G)$, then $\mathcal{S}(G) \cup \{w\}$ is a mutual 3-visibility set, which is a contradiction. Therefore, (ii) follows.

Conversely, if (ii) holds, then by Proposition 4.5 we deduce (i). □

5. Corona and Lexicographic Products

The *corona product* $G \odot H$ is the graph obtained from the disjoint union of G and $\mathrm{n}(G)$ copies of H, denoted by H_i, $i \in \{1, 2, \ldots, \mathrm{n}(G)\}$. If $V(G) = \{u_1, \ldots, u_{\mathrm{n}(G)}\}$, then the corona product $G \odot H$ is constructed by making u_i

adjacent to every vertex in H_i for each $i \in \{1, 2, \ldots, \mathrm{n}(G)\}$. It is readily seen that $K_1 \odot H \cong K_1 + H$.

Proposition 5.1. *The following statements hold for every connected graph G of order $\mathrm{n}(G) \geq 2$ and every positive integer r.*

(i) *If H is a non-complete graph, then $\mu^d(G \odot H) = \mathrm{n}(G)\mathrm{n}(H)$ for every integer $d \geq 2$,*
(ii) $\mu^d(G \odot K_r) = r \cdot \mathrm{n}(G)$ *for every integer $d \geq 3$.*
(iii) $\mu^2(G \odot K_r) = r \cdot \mathrm{n}(G) + \alpha(G)$.

Proof. Let H be a non-complete graph, and let $V(G) = \{u_1, \ldots, u_{\mathrm{n}(G)}\}$ and $d \geq 2$. Notice that for any $u_i \in V(G)$, the subgraph of $G \odot H$ induced by $\{u_i\} \cup V(H_i)$, which is isomorphic to $K_1 + H$, is a convex subgraph. Thus, since $V(G \odot H) = \bigcup_{i=1}^{\mathrm{n}(G)} \{u_i\} \cup V(H_i)$, from Theorem 3.7 (ii) and Proposition 4.1 we deduce that

$$\mu^d(G \odot H) \leq \sum_{i=1}^{\mathrm{n}(G)} \mu^d(K_1 + H) = \mathrm{n}(G)\mathrm{n}(H).$$

To conclude the proof of item (i), we only need to observe that $\bigcup_{i=1}^{\mathrm{n}(G)} V(H_i)$ is a mutual d-visibility set of $G \odot H$, and so

$$\mu^d(G \odot H) \geq \left| \sum_{i=1}^{\mathrm{n}(G)} V(H_i) \right| = \mathrm{n}(G)\mathrm{n}(H).$$

By Proposition 4.6 we derive (ii). Finally, to prove (iii) we first observe that there exists a $\mu^2(G \odot K_r)$-set X such that $\mathcal{S}(G \odot K_r) \subseteq X$, by Lemma 4.3. Hence, there are no adjacent vertices $u_i, u_j \in V(G)$ belonging to X, as otherwise for any simplicial vertex x belonging to the copy of K_r associated to u_i, we have that u_j and x are not $(X, 2)$-visible. Thus, $X \cap V(G)$ is an independent set. Furthermore, for any independent set Y of G we have that $Y \cup (V(G \odot K_r) \setminus V(G)) = Y \cup \mathcal{S}(G \odot K_r)$ is a mutual 2-visibility set of $G \odot K_r$. Therefore, the $\mu^2(G \odot K_r)$-sets have the form $Y \cup (V(G \odot K_r) \setminus V(G))$, where Y is any $\alpha(G)$-set, which implies that $\mu^2(G \odot K_r) = |Y \cup (V(G \odot K_r) \setminus V(G))| = |Y| + r \cdot \mathrm{n}(G) = \alpha(G) + r \cdot \mathrm{n}(G)$. □

Notice that the result above also shows some examples of graphs where the equality holds for Theorem 3.7 (ii).

We next consider the lexicographic product of two graphs and compute the exact value of its mutual d-visibility number in terms of the orders of its factors and the total mutual-visibility number of the first factor. We recall that the *lexicographic product* of two graphs G and H, denoted by $G \circ H$, is a graph with vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices (u, v) and (x, y) are adjacent if $ux \in E(G)$ or ($u = x$ and $vy \in E(H)$). Notice that

$K_2 \circ G \cong G + G$ for any graph G, from which we can say that this product is a kind of generalization of join graphs.

In order to study the mutual d-visibility number of lexicographic products we need to keep two facts in mind. First, a lexicographic product of two graphs is connected if and only if its first factor is connected. Moreover, we should remember the distance formula in lexicographic products. For more information on this product, its properties and applications, we suggest [12].

Remark 5.2 [12]. For a connected non-trivial graph G and two vertices (u, v) and (x, y) of $G \circ H$ we have that

$$\mathrm{d}_{G\circ H}((u,v),(x,y)) = \begin{cases} \mathrm{d}_G(u,x) & \text{if } u \neq x, \\ \min\{\mathrm{d}_H(v,y), 2\} & \text{if } u = x. \end{cases}$$

The following lemma will be one of our main tools.

Lemma 5.3. *Let G be a connected graph of order $\mathrm{n}(G) \geq 2$. For any graph H there exists a $\mu^d(G \circ H)$-set X such that $|\{u\} \times V(H) \cap X| \geq \mathrm{n}(H) - 1$ for every vertex $u \in V(G)$.*

Proof. We can assume $d \geq 2$ and $\mathrm{n}(H) \geq 2$. Let X be a $\mu^d(G \circ H)$-set, $u \in V(G)$ and $X_u = \{u\} \times V(H) \cap X$. Suppose that there exist two different vertices $v, v' \in V(H)$ such that $(u, v), (u, v') \notin X_u$. We now differentiate two cases.

Case 1. $N_G(u) \times V(H) \subseteq X$. For any vertex $(u^*, v^*) \in N_G(u) \times V(H)$ we define the set $X^* = (X \setminus \{(u^*, v^*)\}) \cup \{(u, v')\}$. Notice that for any pair of vertices $(g, h), (x, y) \in X \setminus X_u$ such that (u, v') lies on a shortest path P that makes them mutual (X, d)-visible, there exists the shortest path P', obtained from P by replacing (u, v') by (u, v), which makes them mutual (X^*, d)-visible. Analogously, for any pair of vertices $(u, a), (u, b) \in X_u$ such that the shortest path $(u, a), (u, v'), (u, b)$ makes them mutual (X, d)-visible, there exists the shortest path $(u, a), (u^*, v^*), (u, b)$ which makes them mutual (X^*, d)-visible. Hence, X^* is a $\mu^d(G \circ H)$-set with $|\{u\} \times V(H) \cap X^*| > |X_u|$.

Case 2. $N_G(u) \times V(H) \setminus X \neq \varnothing$. If we use for the set $X'' = X \cup \{(u, v')\}$ arguments similar to those used in Case 1 for X^*, then we can check that X'' is a mutual d-visibility set, which is a contradiction, as $\mu^d(G \circ H) = |X| < |X''|$.

Therefore, either the set X^* satisfies the statement of our lemma or we can repeat the procedure described above until we obtain a $\mu^d(G \circ H)$-set that satisfies the lemma. □

We also need to establish the following two results obtained in [18] where $\gamma(G)$ denotes the *domination number* of G, which is the cardinality of a smallest set S of vertices of G such that any vertex not in S has a neighbor in S.

Proposition 5.4 [18]. *Given a graph G, the following statements hold.*

(i) $\mu_t(G) = \mathrm{n}(G)$ *if and only if* G *is a complete graph.*
(ii) $\mu_t(G) = \mathrm{n}(G)-1$ *if and only if* G *is a non-complete graph with* $\gamma(G) = 1$.

Theorem 5.5 [18]. *Given a connected graph* G *and a non-trivial graph* H, *the following statements hold.*

(i) *If* $\gamma(G) \geq 2$, *then* $\mu_t(G \circ H) = \mathrm{n}(G)(\mathrm{n}(H) - 1) + \mu_t(G)$.
(ii) *If* G *and* H *are not simultaneously complete graphs and* $\gamma(G) = \gamma(H) = 1$, *then* $\mu_t(G \circ H) = \mathrm{n}(G)\mathrm{n}(H) - 1$.
(iii) *If* $\gamma(G) = 1$ *and* $\gamma(H) \geq 2$, *then* $\mu_t(G \circ H) = \mathrm{n}(G)\mathrm{n}(H) - 2$.

We are now ready to present the main result of this section.

Theorem 5.6. *Let* G *be a connected graph of order* $\mathrm{n}(G) \geq 2$ *and* H *a graph of order* $\mathrm{n}(H) \geq 2$. *Then the following statements hold for any integer* $d \geq 2$.

(i) *If* $\gamma(G) \geq 2$, *then* $\mu^d(G \circ H) = \mathrm{n}(G)(\mathrm{n}(H) - 1) + \mu_t(G)$.
(ii) *If* $\gamma(G) = 1$ *and* $H \cong (K_1 \cup K_r) + H'$ *for some non-complete graph* H' *and some non-negative integer* r, *then* $\mu^d(G \circ H) = \mathrm{n}(G)\mathrm{n}(H) - 1$.
(iii) *If* $\gamma(G) = 1$, $\gamma(H) \geq 2$ *and* $H \ncong (K_1 \cup K_r) + H'$ *for every non-complete graph* H' *and every non-negative integer* r, *then* $\mu^d(G \circ H) = \mathrm{n}(G)\mathrm{n}(H) - 2$.

Proof. First, we proceed to prove that $\mu^d(G \circ H) \leq \mathrm{n}(G)(\mathrm{n}(H) - 1) + \mu_t(G)$ for every $d \geq 2$. Let W be a $\mu^d(G \circ H)$-set satisfying Lemma 5.3, and let

$$W_G = \{w \in V(G) : \{w\} \times V(H) \subseteq W\}.$$

We proceed to show that W_G is a total mutual-visibility set of G. Let $x, x' \in V(G)$ be two non-adjacent vertices, and let $x = x_0, \ldots, x_k = x'$ be a shortest path in G. Suppose that $\{x_1, \ldots, x_{k-1}\} \cap W_G \neq \varnothing$. Let $i = \min\{j \in \{1, 2, \ldots, k-1\} : x_j \in W_G\}$. Since vertices in $W \cap \{x_{i-1}\} \times V(H)$ and vertices in $W \cap \{x_{i+1}\} \times V(H)$ are (W, d)-visible, there exists $(z_i, v) \in V(G) \times V(H) \setminus W$ such that x_{i-1}, z_i, x_{i-1} is a shortest path. Thus, $x = x_0, \ldots, x_{i-1} z_i, x_{i+1}, \ldots, x_k = x'$ is a shortest path in G. Now, if $\{x_{i+1}, \ldots, x_{k-1}\} \cap W_G = \varnothing$, then x and y are W_G-visible, otherwise we apply the process described for x and y, starting from z_i and y, until we obtain a shortest path in G which makes W_G-visible the vertices x and y. Hence, W_G is a total mutual-visibility set of G and so $|W_G| \leq \mu_t(G)$. Thus, by Lemma 5.3 we have that

$$\begin{aligned}
\mu^d(G \circ H) &= |W| \\
&= |W_G| \cdot \mathrm{n}(H) + |V(G) \setminus W_G|(\mathrm{n}(H) - 1) \\
&= \mathrm{n}(G)(\mathrm{n}(H) - 1) + |W_G| \\
&\leq \mathrm{n}(G)(\mathrm{n}(H) - 1) + \mu_t(G).
\end{aligned}$$

Therefore,

$$\mu_t(G \circ H) \leq \mu^d(G \circ H) \leq \mathrm{n}(G)(\mathrm{n}(H) - 1) + \mu_t(G) \tag{4}$$

where the first inequality holds because $\mu_t(G') \leq \mu(G')$ for any graph graph G' (see [4]) and by (1). Thus, if $\gamma(G) \geq 2$ we conclude the proof of (i) by Theorem 5.5 (i). Now, if G and H satisfy the assumptions of (iii), then Theorem 5.5 leads to $\mu_t(G \circ H) = \mathrm{n}(G)\mathrm{n}(H) - 2$, and Proposition 5.4 leads to $\mu_t(G) \leq \mathrm{n}(G) - 2$. Thus, by (4) we conclude the proof of (iii).

On the other hand, if $\gamma(G) = 1$ and $H \cong (K_1 \cup K_r) + H'$, for some non-complete graph H' and some non-negative integer r, then $G \circ H \cong (K_1 \cup K_r) + G'$ where G' is a non-complete graph. If u is a universal vertex of G and $V(K_1) = \{v\}$, then for $r = 0$ the subgraph G' is induced by the set $V(G) \times V(H) \setminus \{(u, v)\}$, i.e., (u, v) is a universal vertex, while for $r > 0$, the subgraph G' is induced by $V(G) \times V(H) \setminus \{u\} \times (V(K_r) \cup \{v\})$. Therefore, we conclude that (ii) follows from Propositions 4.1. Observe that the case in which G and H are not simultaneously complete graphs and $\gamma(G) = \gamma(H) = 1$ is indeed included in (ii), as we are assuming $K_1 \cup K_0 \cong K_1$. □

6. Computational Complexity

In this section we discuss the computational complexity of finding the mutual d-visibility number of a graph.

MUTUAL d-VISIBILITY PROBLEM
INSTANCE: A connected non-trivial graph G, an integer $d \geq 2$ and a positive integer $k \leq |V(G)|$.
QUESTION: Is $\mu^d(G) \geq k$?

We should first mention that the MUTUAL d-VISIBILITY PROBLEM is NP-complete for $d = \mathrm{diam}(G)$ [7]. Nevertheless, our reduction presented below works for all cases $d \geq 2$, including $d = \mathrm{diam}(G)$. We will now formally define another known decision problem that we need to continue our studies.

INDEPENDENCE PROBLEM
INSTANCE: A graph G and a positive integer $k \leq |V(G)|$.
QUESTION: Is $\alpha(G) \geq k$?

Theorem 6.1 *[13]. The INDEPENDENCE PROBLEM is NP-complete.*

Notice that Theorem 5.1 (iii) allows to develop a polynomial reduction from the INDEPENDENCE PROBLEM to the MUTUAL d-VISIBILITY PROBLEM when $d = 2$. Thus, the following conclusion is obtained.

Corollary 6.2. *The MUTUAL 2-VISIBILITY PROBLEM is NP-complete.*

We next show that the MUTUAL d-VISIBILITY PROBLEM remains with the same complexity as stated before when $3 \leq d \leq \mathrm{diam}(G) - 1$. To do this we need the following definition. A *rooted graph* is a graph in which one vertex is labelled in a special way in order to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labelled graph on

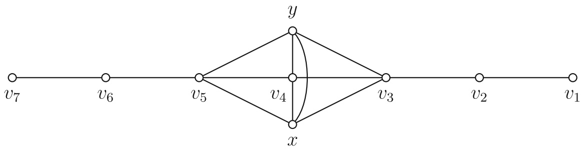

FIGURE 4. The graph H, where $d = 4$

$\mathrm{n}(G) \geq 2$ vertices. Let H_1, H_2,...,$H_{\mathrm{n}(G)}$ be $\mathrm{n}(G)$ copies of a graph H rooted at vertex v. The *rooted product graph* $G \circ_v H$ is the graph obtained by identifying the root of H_i with the i^{th} vertex of G [11]. More formally, assuming that $V(G) = \{u_1, \dots, u_{\mathrm{n}(G)}\}$, we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^{\mathrm{n}(G)} \{(u_i, x)(u_i, y) : \ xy \in E(H)\} \cup \{(u_i, v)(u_j, v) : \ u_i u_j \in E(G)\}.$$

Now, we shall construct a graph G^d in the following way. We consider first a graph H obtained from a path $P_{2d-1} = v_1, \dots, v_{2d-1}$, by adding two extra vertices x, y and the edges xy, xv_{d-1}, xv_d, xv_{d+1}, yv_{d-1}, yv_d and yv_{d+1}. See Figure 4 for an example of a graph H with $d = 4$.

Next, for a given connected graph G with vertex set $V(G) = \{u_1, \dots, u_{\mathrm{n}(G)}\}$ and order $\mathrm{n}(G) \geq 3$, we make the rooted product $G \circ_v H$, where the root v of H is the vertex v_{2d-1}. Finally, to obtain G^d, we add an extra vertex denoted by z and all the edges between z and the vertices of G. As an example, the graph $(C_5)^4$ is represented in Figure 5.

Theorem 6.3. *The MUTUAL d-VISIBILITY PROBLEM is NP-complete, for any $d \geq 3$.*

Proof. First, notice that the MUTUAL d-VISIBILITY PROBLEM is a member of NP since we can check in polynomial time that a given set is indeed a mutual d-visibility set. Now we present a polynomial reduction from the INDEPENDENCE PROBLEM to the MUTUAL d-VISIBILITY PROBLEM. To this end, we consider a graph G and the graph G^d as described above (notice that G^d can be constructed in polynomial time with respect to the order of G). Then, we show that deciding whether $\alpha(G) \geq k$ is equivalent to deciding whether $\mu^d(G^d) \geq k + 4\mathrm{n}(G)$.

Let S be a $\alpha(G)$-set. Let $X \subset V(G^d)$ defined as follows. The set X contains all the vertices of S (vertices (u_i, v) with $u_i \in S$), all the copies of the vertices $x, y, v_1, v_d \in V(H)$ (vertices $(u_i, x), (u_i, y), (u_i, v_1), (u_i, v_d)$ for every $u_i \in V(G)$) of each copy of H in G^d. See bolded vertices in Figure 5. We claim that X is a mutual d-visibility set of G^d.

Clearly, any two vertices of S in G are (X, d)-visible, since they are not adjacent and are at distance two, by using the vertex z, which is not in X.

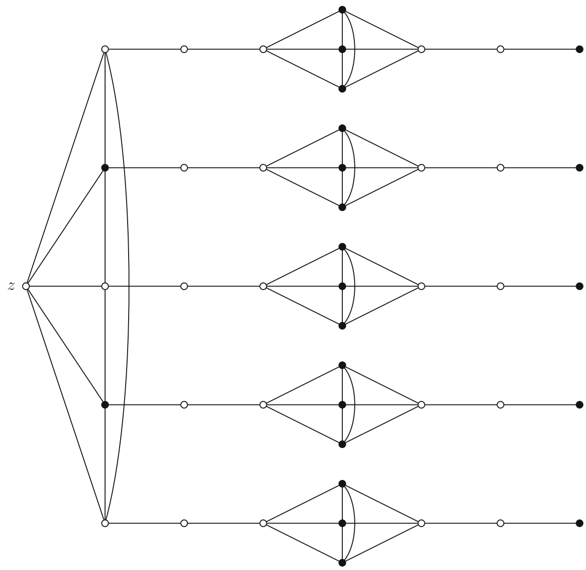

FIGURE 5. The graph G^4, where G is a cycle $C_5 = u_1u_2 \cdots v_5v_1$ and H as previously defined (notice that the root of H is v_7). The bolded vertices form $\mu^4(G^4)$-set

Also, any two vertices of the set $\{(u_i, x), (u_i, y), (u_i, v_d)\}$, with $u_i \in V(G)$, are (X, d)-visible because they form a clique in G^d. Moreover, if $(a, b) \in X$ is one of the pairs of vertices $((u_i, v_1), (u_i, x))$, $((u_i, v_1), (u_i, y))$, $((u_i, v_1), (u_i, v_d))$, $((u_i, x), (u_i, v))$, $((u_i, y), (u_i, v))$, or $((u_i, v_d), (u_i, v))$, where $u_i \in V(G)$, then they are (X, d)-visible, since there is no other vertex of X in the unique shortest a, b-path. Finally, any other two vertices of X are at distance larger than d, and thus they are (X, d)-visible. Consequently, as claimed, X is a mutual d-visibility set of G^d, and so $\mu^d(G^d) \geq 4\mathrm{n}(G) + \alpha(G)$.

On the other hand, let W be a $\mu^d(G^d)$-set. Since the diameter of each copy of H equals $2d-2$, and the fact that $(u_i, x), (u_i, y), (u_i, v_d)$ form a clique, we deduce that each copy of $H - v$ (the subgraph of H obtained by removing the root) in G^d cannot contain more than 4 elements of W. Thus $|W \cap (V(H_i) \setminus (u_i, v))| \leq 4$.

Now, let $W' = W \cap V(G) \times \{v\}$. Suppose W' is not an independent set. Hence, there are two vertices $(u_i, v), (u_j, v) \in W'$ such that they are adjacent. Consider the vertices $(u_i, x), (u_i, y), (u_i, v_d)$ and $(u_j, x), (u_j, y), (u_j, v_d)$. Since the distance between any two vertices (u_i, a) and (u_j, v), and also between (u_j, b) and (u_i, v), where $a, b \in \{x, y, v_d\}$ is exactly d, it follows that $W \cap \{(u_i, x), (u_i, y), (u_i, v_d)\} = \varnothing$ or $W \cap \{(u_j, x), (u_j, y), (u_j, v_d)\} = \varnothing$. Otherwise, there are at least two vertices that are not (W, d)-visible. Assume $W \cap \{(u_i, x), (u_i, y), (u_i, v_d)\} = \varnothing$. We also note that no vertex (u_i, v_ℓ) with $\ell \in$

$\{d+1,\ldots,2d-2\}$ is in W by using the same arguments. In addition, we deduce that it must happen $|W\cap V(H_i)|\leq 3$ (recall $(u_i,v)\in W$), as $\mathrm{diam}(H)=2d-2$ and the vertices $(u_i,v_{d-1}),(u_i,v_{d-2},\ldots,(u_i,v_1))$ form a path of length $d-2$ in H_i. Now, let $W''=(W\setminus(W\cap V(H_i))\cup\{(u_i,x),(u_i,y),(u_i,v_d),(u_i,v_1)\}$. If any two vertices of W were (W,d)-visible, then any two vertices of W'' are also (W'',d)-visible because $(u_i,v)\in W$ and $(u_i,v)\notin W''$ and the remaining vertices added to W'' are at distance larger than d from the remaining vertices of $W\setminus(W\cap V(H_i))$. However, this means that we have constructed a mutual d-visibility set with more vertices than W, which is not possible. Consequently, W' is an independent set of G, and so $\alpha(G)\geq|W'|$. Now, notice that $W'\neq\varnothing$, since $\mu^d(G^d)\geq 4\mathrm{n}(G)+\alpha(G)$, $|W\cap(V(H_i)\setminus(u_i,v))|\leq 4$ and $\alpha(G)\geq 1$ always happens. Suppose next $z\in W$, and consider a vertex $(u_i,v)\in W'$. By using some similar arguments as the ones before, it must happen that $W\cap\{(u_i,x),(u_i,y),(u_i,v_d)\}=\varnothing$. However, we can again construct mutual d-visibility set of G^d with more vertices than W, by removing z from W and adding the vertices $(u_i,x),(u_i,y),(u_i,v_d)$, a contradiction. Thus, z in not in W.

Finally, we have that

$$\mu^d(G^d)=|W|=|W'|+\sum_{i=1}^{\mathrm{n}(G)}|W\cap(V(H_i)\setminus(u_i,v))|\leq\alpha(G)+4\mathrm{n}(G),$$

which leads to the equality $\mu^d(G^d)=4\mathrm{n}(G)+\alpha(G)$. This completes the reduction, and the proof of the NP-completeness of the MUTUAL d-VISIBILITY PROBLEM. □

7. Concluding Remarks

In this article we have studied the total mutual d-visibility problem in graphs, Next, we highlight some problem that can be taken as starting point for further investigation on this concept.

- Study the behaviour of the mutual d-visibility number for the case of Cartesian product graphs, direct product graphs, strong product graphs and generalized Sierpiński graphs.
- Characterize all graphs G with $\mu^d(G)=\mathrm{gp}_d(G)$.
- Characterize all graphs G with $\mu^d(G)=|\mathcal{S}(G)|$.
- Study the parameter $diss_k(G)$.

Acknowledgements

D. Kuziak thanks the Spanish Ministry of Science and Innovation due to grant PID2023-146643NB-I00. This research was made while D. Kuziak was making a stay at the Rovira i Virgili University funded by “Ayudas para la recualificación del sistema universitario español para 2021-2023” programm in the frame of "Marco del Real Decreto 289/2021, de 20 de abril de 2021". L. P. Montejano and J. A. Rodríguez-Velázquez were supported by the SGR Grant 2021 00115, by the project "HERMES" funded by INCIBE and the European Union NextGenerationEU/PRTR, and by the Spanish Ministry of Science and Innovation due to the grants PID2023-146925OB-I00 and PID2023-148202OB-C21.

Funding Funding for open access publishing: Universidad de Cádiz/CBUA

Data Availability Statement Our manuscript has no associated data.

Declarations

Competing interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Aljohani, A., Sharma, G.: Complete visibility for mobile robots with lights tolerating faults. Int. J. Netw. Comput. **8**, 32–52 (2018)

[2] Cicerone, S., Di Stefano, G., Droždek, L., Hedžet, J., Klavžar, S., Yero, I.G.: Variety of mutual-visibility problems in graphs. Theoret. Comput. Sci. **974**, 114096 (2023)

[3] Cicerone, S., Di Stefano, G., Klavžar, S.: On the mutual-visibility in Cartesian products and in triangle-free graphs. Appl. Math. Comput. **438**, 127619 (2023)

[4] Cicerone, S., Di Stefano, G., Klavžar, S., Yero, I.G.: Mutual-visibility in strong products of graphs via total mutual-visibility. Discrete Appl. Math. **358**, 136–146 (2024)

[5] Cicerone, S., Di Stefano, G., Klavžar, S., Yero, I.G.: Mutual-visibility problems on graphs of diameter two. European J. Combin. **120**, 103995 (2024)

[6] Di Luna, G.A., Flocchini, P., Chaudhuri, S.G., Poloni, F., Santoro, N., Viglietta, G.: Mutual visibility by luminous robots without collisions. Inf. Comput. **254**, 392–418 (2017)

[7] Di Stefano, G.: Mutual visibility in graphs. Appl. Math. Comput. **419**, 126850 (2022)

[8] Dudeney, H.E.: Amusements in Mathematics. Nelson, Edinburgh (1917)

[9] Flocchini, P., Prencipe, G., Santoro, N.: Moving and computing models: robots, pp. 3–14. Current Research in Moving and Computing, Distributed Computing by Mobile Entities (2019)

[10] Garey, M.R., Johnson, D.S.: Strong NP-Completeness Results: Motivation, Examples, and Implications. J. ACM. **25**(3), 499–508 (1978)

[11] Godsil, C.D., McKay, B.D.: A new graph product and its spectrum. Bull. Austral. Math. Soc. **18**(1), 21–28 (1978)

[12] Hammack, R., Imrich, W., Klavžar, S.: Handbook of Product Graphs, 2nd edn. CRC Press, Boca Raton, FL (2011)

[13] Karp, R.M.: Reducibility Among Combinatorial Problems. In: Complexity of Computer Computations, R.E. Miller J.W. Thatcher, editors, 85–104, Plenum Press, New York (1972)

[14] Klavžar, S., Kuziak, D., Peterin, I., Yero, I.G.: A Steiner general position problem in graph theory. Comput. Appl. Math. **40**(6), 1–15 (2021)

[15] Klavžar, S., Manuel, P.: A general position problem in graph theory. Bull. Aust. Math. Soc. **98**, 177–187 (2018)

[16] Klavžar, S., Rall, D.F., Yero, I.G.: General d-position sets, Ars Math. Contemp. 21 #P1.03 (2021)

[17] Körner, J.: On the extremal combinatorics of the Hamming space. J. Combin. Theory Ser. A **71**, 112–126 (1995)

[18] Kuziak, D., Rodríguez-Velázquez, J.A.: Total mutual-visibility in graphs with emphasis on lexicographic and Cartesian products. Bull. Malays. Math. Sci. Soc. **46**, 197 (2023)

[19] Meir, A., Moon, J.W.: Relations between packing and covering numbers of a tree. Pacific J. Math. **61**, 225–233 (1975)

[20] Poudel, P., Aljohani, A., Sharma, G.: Fault-tolerant complete visibility for asynchronous robots with lights under one-axis agreement. Theor. Comput. Sci. **850**, 116–134 (2021)

[21] Tian, J., Klavžar, S.: Graphs with total mutual-visibility number zero and total mutual-visibility in Cartesian products. Discuss. Math. Graph Theory **44**, 1277–1291 (2024)

[22] Ullas Chandran, S.V., Parthasarathy, G.J.: The geodesic irredundant sets in graphs. Int. J. Math. Combin. **4**, 135–143 (2016)

[23] Yannakakis, M.: Node-deletion problems on bipartite graphs. SIAM J. Comput. **10**, 310–327 (1981)

Dorota Kuziak
Departamento de Estadística e Investigación Operativa
Universidad de Cádiz
Av. Ramón Puyol s/n
11202 Algeciras
Spain
e-mail: dorota.kuziak@uca.es

Luis P. Montejano and Juan A. Rodríguez-Velázquez
Departament d'Enginyeria Informàtica i Matemàtiques
Universitat Rovira i Virgili
Av Paisos Catalans, 26
43007 Tarragona
Spain
e-mail: luispedro.montejano@urv.cat;
juanalberto.rodriguez@urv.cat

Received: December 1, 2024.
Accepted: May 28, 2025.