Assessing the convergence of three up-and-down moves in binomial option pricing modelling to Black-Scholes-Merton in a fuzzy setting

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Abstract. The fuzzy binomial option pricing (FBOP) method parameterizes the up and down multipliers via volatility, which is assumed to be a fuzzy number, and employs the Cox, Ross, and Rubinstein (CRR) structure for these multipliers. This paper examines alternative parameterizations for the up and down factors within the FBOP framework, specifically the Rendleman and Bartter (RB) and Trigeorgis (TRIG) binomial approximations. We have observed that although all three methodologies produce prices that converge to those of the Black-Scholes-Merton (BSM) model in a fuzzy setting, the RB approach typically results in smaller errors than TRIG and CRR. This superiority is more pronounced for at-the-money options than for other money degrees. While the RB model consistently shows the greatest convergence for both call and put options, its advantage is particularly evident in call options. The superiority of RB is also observed across all volatility scenarios (low, medium, and high). Additionally, Trigeorgis' parameterization often provides a better approximation than CRR does, especially for out-of-the-money options and puts. Therefore, we believe it is worthwhile to explore alternative parameterizations for the up- and down-factors in future FBOP studies as alternatives to CRR.

Keywords: Fuzzy numbers; Fuzzy random option pricing; Fuzzy binomial trees.

1. Introduction

Black and Scholes [6] and Merton [23] introduced the Black-Scholes-Merton option pricing model (BSM), which is widely recognized as the cornerstone of option pricing theory [24] and a groundbreaking innovation for the financial industry [12]. However, while BSM provides a powerful approach for pricing contingent assets, it has limitations in reliably pricing American options, path-dependent options such as barrier options [35] and low-structured real options [34]. As a result, the binomial option approach, which is considered a sister model of BSM [12], has gained significant attention and recognition from practitioners and researchers alike [12,35]. Although the Cox, Ross, and Rubinstein [15] (CRR) method is the most widely used binomial option pricing method, the literature has also proposed at least 10 other binomial methods, such as that by Rendleman and Bartter [31] (RB), which was also developed with the stock market in mind and was published simultaneously to [15], or that by Trigeorgis [34] (TRIG), which was developed in a real option pricing setting.

Financial activities and business management are performed with information that presents different levels of knowledge. Risk implies that the probabilities of the alternative realizations of an event can be stated. On the other hand, uncertainty supposes that the probabilities of possible outcomes of matter cannot be quantified, so no unequivocal statement of them can be obtained [36].

Although many economic phenomena such as option pricing can be modelled with the help of stochastic mathematics, the knowledge of the parameters that regulate probabilistic price movements could be affected by issues such as vagueness or ambiguity. Fuzzy set theory (FST) has provided reliable option pricing instruments for modelling nonprobabilistic uncertainty, such as fuzzy numbers or fuzzy regression [26].

There are numerous examples in which the evaluator may inaccurately perceive the parameters necessary to determine the price of an option. An obvious case is the current price of the underlying asset, which typically, during a trading session, is not traded at a single price but a range of prices [2]. Another example can be seen in the volatility of the underlying asset (σ), which in financial models is often considered a crisp parameter. Using the point estimate of volatility from empirical data (for example, 10%) allows the quantification of σ as a real parameter, but this approach involves losing much information about the possible variability of the parameter. Using all available information in the sample requires the estimation of σ through confidence intervals that can be structured using fuzzy numbers. Thus, suppose that for σ whose point estimate is 10%, the confidence interval with a probability level of 99% is [8%, 12%]. Roughly speaking, the fuzzy modelling of such volatility could be performed with a fuzzy number with a core of 10% and support [8%,12%] (see [13,14,32]).

Probability theory provides rigorous analytical and theoretical grounds. This explains why option pricing formulas have been developed with stochastic mathematics from pioneering studies. On the other hand, the addition of fuzzy tools can enhance results from standard models since they make it possible to introduce alternative additional sources of uncertainty to risk in the existing information, such as imprecision or ambiguity [36]. Fuzzy option pricing has emerged as a burgeoning research field in the 21st century, with several approaches being utilized, including the fuzzy random variable approach, fuzzy integrals, the fuzzy payoff method, and fuzzy expert systems [1]. This increasing interest in fuzzy option pricing can be attributed to the flexibility and applicability of fuzzy set theory in capturing uncertainty and imprecision in financial markets, leading to novel methods for pricing options [1].

This research is centred on the field of fuzzy random option pricing with binomial lattices, i.e., fuzzy binomial option pricing (FBOP). A crucial issue is modelling the so-called up and down multipliers of subjacent asset prices, and the FBOP provides two approaches. One common method consists of supposing that these parameters are estimated directly by experts. An alternative approach consists of modelling "up" and "down" factors as functions of the volatility of the subjacent asset, which is the unanimous way to use the CRR framework. This second approach is explored by comparing the CRR and alternative volatility-parameterized multipliers RB [31] and TRIG [34] and supposing fuzzy volatility. In the literature on fuzzy option pricing, it is generally assumed that the calibration of the up and down moves with fuzzy volatility is performed using the CRR model, without analysing whether this method, among the many methods proposed in the option pricing literature [12], is the most suitable. This research gap motivates our work, which analyses the capability of each proposed calibration method for the up and down moves to fit the BSM formula.

Of course, approaching BSM with the binomial method is of little interest if BSM has straightforward application, as is the case for plain vanilla European option pricing. The interest comes from the fact that the use of binomial option pricing in settings such as American options, path-dependent options or real options is justified. The reason is not only that binomial lattices provide an adaptable framework but also that, in the case of "standard" options, they are essentially equal to BSM [12]. Therefore, the binomial option method is a reliable approximation of BSM in contingent asset settings where the "direct" application of BSM is not possible. Thus, the determination of the best model of the up and down binomial moves will be the one that provides a price closest to the BSM formula in plain vanilla European options with the same maturity and strike price [1,12,18,20].

In summary, the assumption of this paper is that a binomial option model is applied to a valuation problem for options without a closed-form solution, and the best binomial model is the one that provides the prices closest to those given by the BSM model for a plain vanilla European option with the same characteristics (strike price, maturity, etc.). The comparison between binomial models is bounded to those proposed in [15, 31, 34], not to all exposed, for eaxample in [12].

This study is structured as follows. The next section presents the fundamentals of arithmetic with fuzzy numbers and describes the calculation of the BSM model with fuzzy parameters. In the third section, the fuzzy binomial option pricing model is generalized to alternative up and down moves to the CRR approach. Specifically, we suggest using those proposed in [31] and [34] as alternatives. In the final section, in line with comparative studies on the convergence of various binomial models to the BSM under crispness [12,18,20], we conduct a comparative analysis using data from the Spanish financial derivatives market.

2. Fuzzy number arithmetic and the Black–Scholes–Merton formula with fuzzy parameters

A *fuzzy set* $\widetilde{A}$ on a reference set X can be denoted as $\widetilde{A} = \{(x,\mu_A(x))|x \in X\}$, where μ_A is the membership function and a mapping $\mu_A: X\rightarrow[0,1]$. The α-level sets or *α-cuts* of $\widetilde{A}$ are a set $A_\alpha = \{x \in X|\ \mu_{\tilde{A}}(x) \geq \alpha\}, \forall\alpha \in [0,1]$.

A fuzzy number is a fuzzy set $\widetilde{A}$ defined on the reference set $\mathbb{R}$, normal, $\max_{x\in X} \mu_A(x) = 1$, convex, i.e., all its α-cuts are convex and compact sets. Therefore, they can be represented as a set of superposed confidence intervals (α-cuts or α-level sets) $A_\alpha = \left[\underline{A_\alpha},\overline{A_\alpha}\right]$, where $\underline{A_\alpha}$ ($\overline{A_\alpha}$) are continuously increasing (decreasing) functions of α. An FN can be interpreted as a fuzzy quantity approximately equal to the set of real numbers for which the membership function takes a value of 1, A_1.

The *expected value* of FN $\widetilde{A}$, $E(\widetilde{A})$, is a representative real value such that [11]:

$$E(\widetilde{A}) = \frac{1}{2}\left(\int_0^1 \underline{A_\alpha}d\alpha + \int_0^1 \overline{A_\alpha}d\alpha\right). \quad (1a)$$

Let be the FNs $\widetilde{A}$ and $\widetilde{A^*}$. The Euclidean distance between $\tilde{A}$ and $\widetilde{A^*}$, $d(\widetilde{A},\widetilde{A^*})$ is:

$$d(\widetilde{A},\widetilde{A^*}) = \sqrt{\int_0^1 \left(\underline{A_\alpha} - \underline{A_\alpha^*}\right)^2 d\alpha + \int_0^1 \left(\overline{A_\alpha} - \overline{A_\alpha^*}\right)^2 d\alpha}. \quad (1b)$$

Suppose a real valued function $f(x_1,x_2,\ldots,x_n)$ that is increasing with respect to the first m variables $x_j, j = 1,2,\ldots,m$ and decreasing with respect to the other $n-m$ ones, $x_j,\ j = m+1,m+2,\ldots,n$. If we evaluate this function in $x_j = \tilde{A}_j$, j=1,2,…,n, f induces an FN, $\tilde{B}= f(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)$ whose α-cuts, B_α are [8]:

$$B_\alpha = \left[\underline{B_\alpha},\overline{B_\alpha}\right] = \left[f\left(\underline{A_{1\alpha}},\underline{A_{2\alpha}},\ldots,\underline{A_{m\alpha}},\overline{A_{(m+1)\alpha}},\overline{A_{(m+2)\alpha}},\ldots\overline{A_{n\alpha}}\right), f\left(\overline{A_{1\alpha}},\overline{A_{2\alpha}},\ldots\overline{A_{m\alpha}},\underline{A_{(m+1)\alpha}},\underline{A_{(m+2)\alpha}},\ldots,\underline{A_{n\alpha}}\right)\right] \quad (2)$$

Let be a call European option on an asset whose price is S, strike price K, volatility of σ that can be exercised in T years. The free-risk rate is denoted as r. BSM model models the price of the call European option as a function $C(S,K,r,\sigma,T)$:

$$C(S,K,r,\sigma,T) = S\Phi\left(\frac{ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - e^{-rT}K\Phi\left(\frac{ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad (3)$$

where $\Phi(\cdot)$ is the cumulative standard Gaussian function.

Let us suppose that the five parameters of Eq. (3) are fuzzy numbers. Therefore, we symbolize the price as $\tilde{S}$; the strike price as $\widetilde{K}$; the risk-free rate with $\tilde{r}$; volatility as $\widetilde{\sigma}$ and expiration date $\tilde{T}$ as their α-cuts: $S_\alpha = \left[\underline{S_\alpha},\overline{S_\alpha}\right]$, $K_\alpha = \left[\underline{K_\alpha},\overline{K_\alpha}\right]$, $r_\alpha = \left[\underline{r_\alpha},\overline{r_\alpha}\right]$, $\sigma_\alpha = \left[\underline{\sigma_\alpha},\overline{\sigma_\alpha}\right]$ and $T_\alpha = \left[\underline{T_\alpha},\overline{T_\alpha}\right]$, respectively. In this case, Eq. (3) induces a fuzzy price of call options $\tilde{C} = \left(\tilde{S},\widetilde{K},\tilde{r},\widetilde{\sigma},\tilde{T}\right)$ [2,10,13,30,38]. Thus, under the hypothesis $\underline{S_{0_\alpha}} \geq 0$, $\underline{K_\alpha} \geq 0$ the α-cuts $C_{0_\alpha} = \left[\underline{C_\alpha},\overline{C_\alpha}\right]$, are evaluated from Eq. (3) as:

$$C_\alpha = \{x|x = C(S,K,r,\sigma,T), S\in S_\alpha, K\in K_\alpha, r\in r_\alpha, \sigma\in\sigma_\alpha, T\in T_\alpha\}, \quad (4)$$

and considering that $\frac{\partial C}{\partial S} \geq 0, \frac{\partial C}{\partial K} \leq 0, \frac{\partial C}{\partial r} \geq 0, \frac{\partial C}{\partial \sigma} \geq 0$ and $\frac{\partial C}{\partial T} \geq 0$ [1], then with the rule of Eq (2):

$$C_\alpha = \left[\underline{C_\alpha},\overline{C_\alpha}\right] = \left[C\left(\underline{S_\alpha},\overline{K_\alpha},\underline{r_\alpha},\underline{\sigma_\alpha},\underline{T_\alpha}\right), C\left(\overline{S_\alpha},\underline{K_\alpha},\overline{r_\alpha},\overline{\sigma_\alpha},\overline{T_\alpha}\right)\right]. \quad (5)$$

We can proceed analogously in the case of put options. Their price, P, also depends on S,K,r,σ and T in such a way that:

$$P(S,K,r,\sigma,T) = e^{-rT}K\Phi\left(-\frac{ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S\Phi\left(-\frac{ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad (6)$$

In the case of having fuzzy estimates of the parameters $\tilde{S},\widetilde{K},\tilde{r},\widetilde{\sigma}$ and $\tilde{T}$, Eq. (5) induces a fuzzy price of the put option, $\tilde{P} = P\left(\tilde{S},\widetilde{K},\tilde{r},\widetilde{\sigma},\tilde{T}\right)$. Therefore, P_α is:

$$P_\alpha = \{x|x = P(S,K,r,\sigma,T), S\in S_\alpha, K\in K_\alpha, r\in r_\alpha, \sigma\in\sigma_\alpha, T\in T_\alpha\},$$

and considering that $\frac{\partial P}{\partial S} \leq 0, \frac{\partial P}{\partial K} \geq 0, \frac{\partial P}{\partial r} \leq 0, \frac{\partial P}{\partial \sigma} \geq 0$ but $\frac{\partial P}{\partial T}$ can be >0 and <0 [1], by rule in Eq (2), then $P_\alpha = \left[\underline{P_\alpha},\overline{P_\alpha}\right]$:

$$\underline{P_\alpha} = \text{minimum}\left\{P\left(\overline{S_\alpha},\underline{K_\alpha},\overline{r_\alpha},\underline{\sigma_\alpha},T\right), \text{ subject to } \underline{T_\alpha} \leq T \leq \overline{T_\alpha}\right\} \quad (7a)$$

$$\overline{P_\alpha} = \text{maximum}\left\{P\left(\underline{S_\alpha},\overline{K_\alpha},\underline{r_\alpha},\overline{\sigma_\alpha},T\right), \text{subject to } \underline{T_\alpha} \leq T \leq \overline{T_\alpha}\right\}. \quad (7b)$$

Example 1

Let us price 0.5-cuts of the prices of call and put options with the following parameters: $S_{0.5}$ =[103.5, 104.5]; $K_{0.5}$ =[108.5,109.5]; $r_{0.5}$ =[0.0055,0.0065]; $\sigma_{0.5}$=[0.375, 0.425]; and $T_{0.5}$=[0.95,1.05].

For a call option, the 0.5-cut of the price $\tilde{C}$ is $C_{0.5} = \left[\underline{C_{0.5}},\overline{C_{0.5}}\right]$ =[12.14, 17.58]. It has been fitted considering Eqs. (3) and (5):

$\underline{C_{0.5}}$=C(103.5,109.5,0.0055,0.375,0.95)=12.14,

$\overline{C_{0.5}}$=C(104.5,108.5,0.0065,0.425,1.05)=17.48.

On the other hand, for the price of a put option $\tilde{P}$, we find that $P_{0.5} = \left[\underline{P_{0.5}},\overline{P_{0.5}}\right]$ =[17.12, 21.11]. This

0.5-cut is obtained via Eqs. (6), (7a), (7b) and the rule in Eq. (2):

$\underline{P_{0.5}}$ =minimum{P(104.5, 108.5, 0.0065, 0.375, T) subject to 0.95≤T≤1.05}=17.12,

$\overline{P_{0.5}}$=maximum {P(103.5, 109.5, 0.0055, 0.425, T) subject to 0.95≤T≤1.05}=21.11.

3. Fuzzy random binomial option pricing

3.1. General settings in fuzzy random binomial option pricing

The formal origin of the binomial option pricing model may be stated in [15,31]. It supposes that the price of the subjacent asset varies in discrete time due to two possible movements: up (rate $u > 1$) and down (rate $0 < d < 1$). If we symbolize the risk-neutral probability for the up movement, p_u, and to attain a declining rate, d, $p_d = 1 - p_u$, and the period of the jump, h years, we obtain:

$$p_u = \frac{e^{rh} - d}{u - d};\ p_d = \frac{u - e^{rh}}{u - d},$$

Let be a call option with maturity $T = n \cdot h$; the price of a European call option C is

$$C(S,K,r,u,d,n,h) = e^{-r \cdot n \cdot h} \sum_{j=0}^{n} \binom{n}{j} {p_u}^j (1 - p_u)^{n-j} \max\{u^j d^{n-j} S - K, 0\}, \quad (8a)$$

and for a put option:

$$P(S,K,r,u,d,n,h) = e^{-r \cdot n \cdot h} \sum_{j=0}^{n} \binom{n}{j} {p_u}^j (1 - p_u)^{n-j} \max\{K - u^j d^{n-j} S, 0\}. \quad (8b)$$

Roughly speaking, fuzzy-binomial options pricing models up (u) and down (d) multipliers as fuzzy numbers. The FBOP literature can be classified according to 4 criteria:

Criterion 1. *Hypotheses about how fuzzy up and down moves are estimated.* According to this criterion, we differentiate between two approaches. The first supposes that $\tilde{u}$ and $\tilde{d}$ are estimated by experts' judgments independently, i.e., $\tilde{u}$ and $\tilde{d}$ are not connected quantifications [16,21,22,27,33,37] or, alternatively, fit a symmetric increase/decrease rate $\tilde{\alpha}$ in such a way that $\tilde{u} = 1 + \tilde{\alpha}$ and $\tilde{d} = 1 - \tilde{\alpha}$ [9,10]. In this last case, $\tilde{u}$ and $\tilde{d}$ are connected by the rate $\tilde{\alpha}$. The alternative is to link up and down multipliers to the annual subjacent asset volatility, $\tilde{\sigma}$, with the CRR formulation [3,4,17,19,25,29,39,41-45]. Therefore, $\tilde{u} = e^{\tilde{\sigma}\sqrt{h}}$, $\tilde{d} = e^{-\tilde{\sigma}\sqrt{h}}$ and h represent the crisp periodicity of the jumps. Note that in this case, both the up and down moves are functions of the annual volatility $\tilde{\sigma}$, i.e., fitting these multipliers is enough to obtain an estimate of $\tilde{\sigma}$.

Criterion 2. *The parameters assumed to be fuzzy.* The most common approach assumes that only the up and down moves are fuzzy, which implies that their probabilities are also fuzzy. On the other hand, the remaining parameters are considered crisp [1,3,4,16,19,25,27,29,33,39,41,42,43]. However, many works assume that other parameters, such as the risk-free interest rate or initial asset price, are fuzzy [9,10,14,21,22,39,44,45].

Criterion 3. *The types of options being evaluated* The majority of works evaluate European-style options [1,10,19,21,27,29,37,42] or American-style options [22-25,41,44,45]. Additionally, most works explicitly or implicitly focus on options on stocks or stock indices [1,10,19,21,22,27,29,37,42], as well as real options [3,4,14,16,17,33,43,45]. However, the literature also includes other applications, such as vulnerable options [39], guaranteed insurance and annuities [3,4], and business valuations.

Criterion 4. *The shape of the fuzzy numbers that quantify uncertain parameters.* The mainstream approach in these papers uses triangular-shaped fuzzy numbers [3,9,10,16,17,19,27, 29,39,41,43] or trapezoidal numbers [21,25,37,44,45] to model the embedded parameters. Less commonly, adaptive fuzzy numbers [4], octagonal fuzzy numbers [22], parabolic fuzzy numbers [42], and empirically fitted fuzzy numbers [14] have been used. Alternative uncertainty modelling techniques, such as intuitionistic fuzzy numbers or Type-2 fuzzy numbers, have rarely been employed. In the FBOP literature, we can outline [1].

3.2. Fuzzy random binomial option pricing with alternative up and down modelling

Like [9,10], we also allow the price of the subjacent asset to be uncertain by quantifying it with the fuzzy number $\tilde{S}$, the strike price $\tilde{K}$ and the risk-free interest rate $\tilde{r}$. As in all the reviewed literature, we assume that the maturity $T = n \cdot h$ and period h are crisp parameters. The α-cuts of $\tilde{C}$ and $C_\alpha = \left[\underline{C_\alpha}, \overline{C_\alpha}\right]$ are solved via two programming models that consider $\frac{\partial C}{\partial S} \geq 0, \frac{\partial C}{\partial K} \leq 0, \frac{\partial C}{\partial r} \geq 0$ and apply Eq. (2) in Eq. (8a):

$$\underline{C_\alpha} = \text{minimum}\, C\left(\underline{S_\alpha}, \overline{K_\alpha}, \underline{r_\alpha}, d, u\right),$$

$$\text{subject to } \underline{d_\alpha} \le d \le \overline{d_\alpha}, \underline{u_\alpha} \le u \le \overline{u_\alpha}, \quad (9a)$$

$$\overline{C_\alpha} = \text{maximum}\, C\left(\overline{S_\alpha}, \underline{K_\alpha}, \overline{r_\alpha}, d, u\right)$$

$$\text{subject to } \underline{d_\alpha} \le d \le \overline{d_\alpha}, \underline{u_\alpha} \le u \le \overline{u_\alpha}, \quad (9b)$$

where $u_\alpha = \left[\underline{u_\alpha}, \overline{u_\alpha}\right]$ represents the α-levels of $\tilde{u}$ and where $d_\alpha = \left[\underline{d_\alpha}, \overline{d_\alpha}\right]$ corresponds to those of $\tilde{d}$. Similarly, we suppose that $\overline{u_\alpha} \ge e^{\overline{r_\alpha}T} \ge \overline{d_\alpha}$ and that $\underline{u_\alpha} \ge e^{\underline{r_\alpha}T} \ge \underline{d_\alpha}$.

We proceed analogously to obtain the fuzzy price of calls $\tilde{P}$ and $P_\alpha = \left[\underline{P_\alpha}, \overline{P_\alpha}\right]$ by taking into account $\frac{\partial P}{\partial S} \le 0, \frac{\partial P}{\partial K} \ge 0, \frac{\partial P}{\partial r} \le 0$ and applying Eq. (2) in Eq. (8b):

$$\underline{P_\alpha} = \text{minimum}\, P\left(\overline{S_\alpha}, \underline{K_\alpha}, \overline{r_\alpha}, d, u\right),$$

$$\text{subject to } \underline{d_\alpha} \le d \le \overline{d_\alpha}, \underline{u_\alpha} \le u \le \overline{u_\alpha}, \quad (10a)$$

$$\overline{P_\alpha} = \text{maximum}\, P\left(\underline{S_\alpha}, \overline{K_\alpha}, \underline{r_\alpha}, d, u\right)$$

$$\text{subject to } \underline{d_\alpha} \le d \le \overline{d_\alpha}, \underline{u_\alpha} \le u \le \overline{u_\alpha}, \quad (10b)$$

One of the main strengths of binomial option pricing models is that they are flexible enough to model a great variety of optionality situations and features but also tend toward BSM when $h \to 0$, i.e., $n \to \infty$. That is, binomial option pricing allows the use of BSM in cases in which the continuous time option pricing model cannot be applied owing to the rigidity of the formula [12]. To ensure that the convergence literature parameterizes up and down multipliers and their probabilities with respect to subjacent asset volatility σ and the free discount rate r. Therefore, these parameters can be denoted as the functions $u(r,\sigma)$, $d(r,\sigma)$, $p_u(r,\sigma)$, and $p_d(r,\sigma)$. Table 1 displays how these functions have been modelled by CRR, RB and TRIG to approach BSM.

Table 1. Alternative specifications for up and down multipliers and risk-neutral probabilities

	CRR [15]	RB [31]	TRIG [34]
$u(r,\sigma)$	$e^{\sigma\sqrt{h}}$	$e^{(r-\sigma^2/2)h+\sigma\sqrt{h}}$	$e^{\sqrt{\sigma^2 h+(r-\sigma^2/2)^2h^2}}$
$d(r,\sigma)$	$e^{-\sigma\sqrt{h}}$	$e^{(r-\sigma^2/2)h-\sigma\sqrt{h}}$	$e^{-\sqrt{\sigma^2 h+(r-\sigma^2/2)^2h^2}}$
$p_u(r,\sigma)$	$\frac{e^{rh}-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}$	$\frac{e^{\sigma^2h/2}-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}$	$\frac{1}{2}+\frac{1}{2}\frac{(r-\sigma^2/2)h}{\sigma^2h+(r-\sigma^2/2)^2h^2}$
$p_d(r,\sigma)$	$\frac{e^{\sigma\sqrt{h}}-e^{rh}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}$	$\frac{e^{\sigma\sqrt{h}}-e^{\sigma^2h/2}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}$	$\frac{1}{2}-\frac{1}{2}\frac{(r-\sigma^2/2)h}{\sigma^2h+(r-\sigma^2/2)^2h^2}$

In this regard, u, d, p_u and p_d are functions of r and σ, , and consequently, the prices of call and put options turn into functions $C(S,K,r,\sigma)$ and $P(S,K,r,\sigma)$,, respectively. From Eq. (8a), we find for call options:

$$C(S,K,r,\sigma) = e^{-r\cdot n\cdot h}\sum_{j=0}^{n}\binom{n}{j}p_u(r,\sigma)^j p_d(r,\sigma)^{n-j}\max\{u(r,\sigma)^j d(r,\sigma)^{n-j}S - K, 0\}, \quad (11a)$$

From Eq. (8b), we find for put options:

$$P(S,K,r,\sigma) = e^{-r\cdot n\cdot h}\sum_{j=0}^{n}\binom{n}{j}p_u(r,\sigma)^j p_d(r,\sigma)^{n-j}\max\{K - u(r,\sigma)^j d(r,\sigma)^{n-j}S, 0\}, \quad (11b)$$

To evaluate $C(\tilde{S},\tilde{K},\tilde{r},\tilde{\sigma})$ again, it must be accomplished that $\frac{\partial C}{\partial \sigma} \ge 0$. Therefore, for any parameterization of Table 7, the α-cuts of the call option, $C_\alpha = \left[\underline{C_\alpha}, \overline{C_\alpha}\right]$, are obtained from taking into account Eqs. (9a), (9b) and (11a):

$$C_\alpha = \left[\underline{C_\alpha}, \overline{C_\alpha}\right] = \left[C\left(\underline{S_\alpha}, \overline{K_\alpha}, \underline{r_\alpha}, \underline{\sigma_\alpha}\right), C\left(\overline{S_\alpha}, \underline{K_\alpha}, \overline{r_\alpha}, \overline{\sigma_\alpha}\right)\right]. \quad (12a)$$

By following a similar argument, for the put price, $P(\tilde{S},\tilde{K},\tilde{r},\tilde{\sigma})$, by using Eqs. (10a), (10b) and (11b):

$$P_\alpha = \left[\underline{P_\alpha}, \overline{P_\alpha}\right] = \left[P\left(\overline{S_\alpha}, \underline{K_\alpha}, \overline{r_\alpha}, \underline{\sigma_\alpha}\right), P\left(\underline{S_\alpha}, \overline{K_\alpha}, \underline{r_\alpha}, \overline{\sigma_\alpha}\right)\right]. \quad (12b)$$

Example 2

Let us price 0.5-cuts of the prices of call and put options with the following parameters: $S_{0.5}$ =[103.5, 104.5]; $K_{0.5}$ =[108.5, 109.5]; $r_{0.5}$ =[0.0055, 0.0065]; $\sigma_{0.5}$=[0.375, 0.425]; and $T = 1$. For the jumps, we suppose h=1/4 and then n=4. The up and down moves are modelled with the CRRs [15] shown in Table 1.

For a call option, the 0.5-cut of the price $\tilde{C}$ is $C_{0.5} = \left[\underline{C_{0.5}}, \overline{C_{0.5}}\right]$ =[14.42, 16.85]. It has been fitted from Eqs. (11a) and (12a) and models u, d, p_u and p_d, as shown in Table 1. Then:

$\underline{C_{0.5}}$ = C(103.5,109.5,0.0055,0.375,0.95)=14.42,

$\overline{C_{0.5}}$ = C(104.5,108.5,0.0065,0.425,1.05)=16.85.

On the other hand, the put option is obtained as $P_{0.5} = \left[\underline{P_{0.5}}, \overline{P_{0.5}}\right]$ =[17.11, 19.87]. This 0.5-cut is obtained by using Eqs. (11b) and (12b), and the moves and probabilities shown in Table 1.

$\underline{P_{0.5}}$ = P(104.5,108.5,0.0065,0.375)=17.11,

$\overline{P_{0.5}} = P(103.5,109.5,0.0055,0.425)=19.87$.

4. Comparing alternative parameterizations of up and down moves of binomial option pricing in fuzzy setting pricing

4.1. Database and general considerations

The parameterization of u and d by the CRR has been widely adopted in the FBOP literature, with $\tilde{u}$ and $\tilde{d}$ being explicitly linked to the volatility of the underlying asset. However, as noted in [44], alternative approaches, as shown in Table 2, could also be considered. The ability of binomial lattices to produce option prices similar to those of the BSM is a key reason for their popularity. This motivates the empirical analysis in this section, where we perform a sensitivity analysis of the convergence of the CRR, RB, and TRIG binomial models (see Table 1) to BSM prices in a fuzzy setting.

Our empirical application is developed with European-style options and with data from the Spanish derivatives market MEFFSA. To develop the sensitivity analysis, we need to state several scenarios for volatility. To do so, we consider the historical volatility of the average daily value of the index on the futures of the IBEX-35 from January 27, 2011, to January 27, 2023, whose basic descriptive statistics are shown in Table 2. This index is the subjacent asset of the options over the IBEX-35 negotiated in the MEFFSA. This represents a broader time period than most empirical studies in a fuzzy option pricing setting [2,19,28,29,42], where it is common for the observations used to span no more than a year. Considering 12 years of analysis will allow us to capture virtually all possible volatility scenarios in the index options market under study.

Table 2. Descriptive statistics of the index of futures on the IBEX 35 from January 27, 2011, to January 27, 2023

	Index	Daily logarithmic growth	Daily 60-day volatility	Annualized 60-day volatility
minimum	6002	-0.13213	0.00548	0.08706
maximum	11784	0.05463	0.03070	0.48737
mean	9078.97	$-4.80\cdot10^{-05}$	0.01086	0.17232

We also perform a sensitivity analysis with respect to the degree of moneyness, considering K=0.9, 1, and 1.1 and differentiating between call and put options. As is common in the FBOP, the rest of the parameters are crisp: T=1, S=1, and r=0%.

We also conduct a sensitivity analysis of the accuracy of the binomial pricing models with respect to the extension of the move periodicity (h). It is considered to be annual (h=1), semiannual (h=1/2), quarterly (h=1/4), monthly (h=1/12), weekly (h=1/48), daily (h=1/252) or 12 hours (h=1/504).

The use of a null risk-free rate has a twofold justification. First, during a great part of the period of reference, the average risk-free rate in the European Union's financial markets is approximately null. Likewise, in the Spanish derivatives market, the options for the IBEX-35 are actually for the futures on that index, which is considered the subjacent asset considered in this empirical application. Therefore, it seems logical to use r=0%, as stated by the extension of the BSM to options on futures by Black [5].

The use of the index futures index, rather than the index related to stocks, allows for a more realistic estimation of the prices and volatilities of IBEX-35 futures options traded in the Spanish market [2]. Similarly, the choice of this index as representative for the Spanish market aligns with the empirical works of FBOP. For instance, [19] uses the S&P 500 index as a representative of the underlying asset prices in the U.S. market, whereas [42] uses the S&P 100 index for the same purpose. On the other hand, [29] employs the DAX index for FBOP analysis in the context of the German options market, and [28] uses the MIBO index for the Italian market. In all the cases, these indices play a role in their respective markets analogous to that of the IBEX-35 futures index in Spain.

Afterwards, we conduct an empirical experiment by estimating fuzzy volatility by superimposing its statistical confidence intervals at multiple significance levels, following the approach proposed in [7,32] and applied in fuzzy random option pricing [1,13,14]. From that fuzzy volatility and taking into account that r=0%, CRR, RB and TRIG moves and probabilities (see Table 1) are adjusted.

4.2. Methodology used to make the comparative assessment of binomial moves modelling

This sensitivity analysis is carried out by implementing the steps below and is performed using a worksheet.

Step 1. The historical volatility in t is calculated as the standard deviation of the logarithmic difference of the index between two sessions in the last 60 days, i.e.,

$$s_t^d = \sqrt{\frac{\sum_{j=t-60}^{t-1}\left[\ln\left(\frac{I_{j+1}}{I_j}\right)-\frac{\sum_{j=t-60}^{t-1}\ln\left(\frac{I_{j+1}}{I_j}\right)}{60}\right]^2}{59}}. \quad (13)$$

Step 2. For the set of historical volatilities t=1,2,…,3019, the percentiles are ε=0.01, 0.5, 0.25, 0.4, 0.5, 0.6, 0.75, 0.95, and 0.99. For the εth percentile, the standard deviation is s_ε^d, which on an annual basis we symbolize as $s_\varepsilon = \sqrt{252} \cdot s_\varepsilon^d$. That is, excessively low volatilities (below the 1st percentile) or excessively high volatilities (above the 99th percentile) are discarded. This step helps to clean the data by removing outliers.

Step 3. The fuzzy daily variance $\left(\widetilde{\sigma_\varepsilon^d}\right)^2$ and the annual $(\widetilde{\sigma_\varepsilon})^2$ are induced by superimposing the statistical confidence intervals that we can construct from $(s_\varepsilon^d)^2$ in Eq. (13). To do so, we consider methodology [7,32]. Therefore,

$$(\sigma_\varepsilon^d)^2_\alpha = \left[\underline{(\sigma_\varepsilon^d)^2_\alpha}, \overline{(\sigma_\varepsilon^d)^2_\alpha}\ \right] = \left[\frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;1-h(\alpha)}}, \frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;h(\alpha)}}\right], \qquad (14)$$

where $h(\alpha)$ is the function that links the membership level with the statistical significance linked to the confidence intervals.

Step 4. We use the linear transformation [32], $h(\alpha) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\alpha + \frac{\gamma}{2}$, where γ is the statistical significance of the confidence interval that serves as support for $\left(\widetilde{\sigma_\varepsilon^d}\right)^2$. In this paper, we fix $\gamma = 0.01$, so the α-levels of $\left(\widetilde{\sigma_\varepsilon^d}\right)^2$ and $(\sigma_\varepsilon^d)^2_\alpha$ in Eq. (14) become:

$$(\sigma_\varepsilon^d)^2_\alpha = \left[\underline{(\sigma_\varepsilon^d)^2_\alpha}, \overline{(\sigma_\varepsilon^d)^2_\alpha}\ \right] = \left[\frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;1-h(\alpha)}}, \frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;h(\alpha)}}\right] =$$

$$= \left[\frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;0.495\alpha+0.005}}, \frac{59\cdot(s_\varepsilon^d)^2}{\chi^2_{59;0.995-0.495\alpha}}\right]. \qquad (15)$$

Setting a lower value for γ will result in fuzzy volatilities with wider support, and consequently, the option prices will be more uncertain.

Step 5. The α-cuts of the fuzzy annual volatility, $(\widetilde{\sigma_\varepsilon})^2$, are calculated via Eq. (15) $(\sigma_\varepsilon)^2_\alpha = 252 \cdot (\sigma_\varepsilon^d)^2_\alpha$:

$$(\sigma_\varepsilon)^2_\alpha = \left[\underline{(\sigma_\varepsilon)^2_\alpha}, \overline{(\sigma_\varepsilon)^2_\alpha}\ \right] = \left[252\cdot\underline{(\sigma_\varepsilon^d)^2_\alpha}, 252\cdot\overline{(\sigma_\varepsilon^d)^2_\alpha}\ \right]. (16)$$

Step 6. The annual historical standard deviation linked to the εth percentile $\widetilde{\sigma_\varepsilon}$ throughout its α-level sets $\sigma_{\varepsilon_\alpha}$ is obtained from Eq. (16) via $\sigma_{\varepsilon_\alpha} = \sqrt{(\sigma_\varepsilon)^2_\alpha}$. Thus,

$$\sigma_{\varepsilon_\alpha} = \left[\underline{\sigma_{\varepsilon_\alpha}}, \overline{\sigma_{\varepsilon_\alpha}}\ \right] = \left[\sqrt{252}\cdot\sqrt{\underline{(\sigma_\varepsilon^d)^2_\alpha}}, \sqrt{252}\cdot\sqrt{\overline{(\sigma_\varepsilon^d)^2_\alpha}}\ \right]. \qquad (17)$$

As a summary of steps 1-6, Table 3 shows the standard deviations used to evaluate the sensitivity of convergence to the BSM by alternative fuzzy specifications of $\tilde{u}$ and $\tilde{d}$.

Table 3. Volatility scenarios generated from the IBEX-35 futures index used in our numerical application

ε	σ_{ε_0} (support of $\widetilde{\sigma_\varepsilon}$)	σ_{ε_1} (core of $\widetilde{\sigma_\varepsilon}$)
0.01	[0.081086, 0.110009]	0.093191
0.05	[0.085956, 0.116616]	0.098788
0.25	[0.101810, 0.138125]	0.117009
0.4	[0.116925, 0.158631]	0.134380
0.5	[0.128603, 0.174475]	0.147802
0.6	[0.143827, 0.195129]	0.165298
0.75	[0.170302, 0.231048]	0.195726
0.95	[0.289275, 0.392458]	0.332459
0.99	[0.416551, 0.565133]	0.478736

Step 7. Fit for every ε and strike price K the price with the fuzzy BSM of call and put options. This fuzzy number is symbolized as $\widetilde{C_{K,\varepsilon}^{BSM}}$ for call options, and its alpha cuts $\left(C_{K,\varepsilon}^{BSM}\right)_\alpha = \left[\underline{\left(C_{K,\varepsilon}^{BSM}\right)_\alpha}, \overline{\left(C_{K,\varepsilon}^{BSM}\right)_\alpha}\right]$ are obtained from Eqs. (4) and (5). Analogously, we can proceed for the put options in such a way that the price of the put option may be symbolized as $\widetilde{P_{K,\varepsilon}^{BSM}}$, whose α-cuts $\left(P_{K,\varepsilon}^{BSM}\right)_\alpha = \left[\underline{\left(P_{K,\varepsilon}^{BSM}\right)_\alpha}, \overline{\left(P_{K,\varepsilon}^{BSM}\right)_\alpha}\right]$ are obtained from Eqs. (6) and (7a--7b). In this step, we have 27 fuzzy prices of call and put options.

Step 8. Fit for every volatility scenario ε, strike price K, and period h the price of options by the three alternative binomial models in Table 1. This implies the use of Eqs. (11a) and (12a) for call options and Eqs. (11b) and (12b) in the case of put options. For the call option that uses the ith binomial approach (Bin_i), we calculate the price $\widetilde{C_{K,\varepsilon,h}^{Bin_i}}$ as:

$$\widetilde{C_{K,\varepsilon,h}^{Bin_i}} = \sum_{j=0}^{n}\binom{n}{j} p_u(0,\widetilde{\sigma_\varepsilon})^j p_d(0,\widetilde{\sigma_\varepsilon})^{n-j}\max\{u(0,\widetilde{\sigma_\varepsilon})^j d(0,\widetilde{\sigma_\varepsilon})^{n-j} - K, 0\}, \qquad (18)$$

where $n = 1/h$ and the probabilities and up and down moves are obtained by applying the results in Table 1. The α-cuts of Eq. (18) and $\left(C_{K,\varepsilon,h}^{Bin_i}\right)_\alpha = \left[\underline{\left(C_{K,\varepsilon,h}^{Bin_i}\right)_\alpha}, \overline{\left(C_{K,\varepsilon,h}^{Bin_i}\right)_\alpha}\right]$ are obtained by applying Eqs. (11a) and (12a) to implement Eq. (18). Similarly, we proceed for the put options in such a way that for a given ε and strike price K, period h, and using the ith binomial approach Bin_i, we obtain $\widetilde{P_{K,\varepsilon,h}^{Bin_i}}$:

$$\widetilde{P_{K,\varepsilon,h}^{Bin_i}} = \sum_{j=0}^{n}\binom{n}{j} p_u(0,\widetilde{\sigma_\varepsilon})^j p_d(0,\widetilde{\sigma_\varepsilon})^{n-j}\max\{K - u(0,\widetilde{\sigma_\varepsilon})^j d(0,\widetilde{\sigma_\varepsilon})^{n-j}, 0\}, \qquad (19)$$

and $\left(P_{K,\varepsilon,h}^{Bin_i}\right)_\alpha = \left[\underline{\left(P_{K,\varepsilon,h}^{Bin_i}\right)_\alpha}, \overline{\left(P_{K,\varepsilon,h}^{Bin_i}\right)_\alpha}\right]$ comes by using Eqs. (11b) and (12b) to evaluate (19).

Step 9. Calculate for every binomial model its distance to the BSM price. To do it we evaluate the rate of the Euclidean distance between $\widetilde{C_{K,\varepsilon}^{BSM}}$ and $\widetilde{C_{K,\varepsilon,h}^{Bin_i}}$ and the expected value of the fuzzy BSM price. These calculations need using $\left(C_{K,\varepsilon}^{BSM}\right)_\alpha$ obtained in step 7 and $\left(C_{K,\varepsilon,h}^{Bin_i}\right)_\alpha$ discussed in step 8. The error by the *i*th binomial approach for a call option with a strike price *K*, in the *ε*th volatility scenario and a period h, $error_{K,\varepsilon,h}^{Bin_i}$, is:

$$error_{K,\varepsilon,h}^{Bin_i} = \frac{d\left(\widetilde{C_{K,\varepsilon}^{BSM}},\widetilde{C_{K,\varepsilon,h}^{Bin_i}}\right)}{E\left(\widetilde{C_{K,\varepsilon}^{BSM}}\right)}, \quad (20)$$

where $d\left(\widetilde{C_{K,\varepsilon}^{BSM}},\widetilde{C_{K,\varepsilon,h}^{Bin_i}}\right)$ was defined in Eq. (1b) and $E\left(\widetilde{C_{K,\varepsilon}^{BSM}}\right)$ in Eq. (1a).

Analogously we proceed with the put options, i.e., the error in put options, $error_{K,\varepsilon,h}^{Bin_i}$, is:

$$error_{K,\varepsilon,h}^{Bin_i} = \frac{d\left(\widetilde{P_{K,\varepsilon}^{BSM}},\widetilde{P_{K,\varepsilon,h}^{Bin_i}}\right)}{E\left(\widetilde{P_{K,\varepsilon}^{BSM}}\right)}, \quad (21)$$

so that Eq. (21) is adjusted using $\left(P_{K,\varepsilon}^{BSM}\right)_\alpha$ obtained in step 7 and $\left(C_{K,\varepsilon,h}^{Bin_i}\right)_\alpha$ adjusted in step 8.

In this step it is expected that the price obtained with any fuzzy binomial approach will tend to be that provided by BSM, i.e., $\lim_{h\to 0} error_{K,\varepsilon,h}^{Bin_i} = 0$. The values of the integrals that are solved to obtain $d\left(\widetilde{C_{K,\varepsilon}^{BSM}},\widetilde{C_{K,\varepsilon,h}^{Bin_i}}\right)$, $d\left(\widetilde{P_{K,\varepsilon}^{BSM}},\widetilde{P_{K,\varepsilon,h}^{Bin_i}}\right)$, $E\left(\widetilde{C_{K,\varepsilon}^{BSM}}\right)$ and $E\left(\widetilde{P_{K,\varepsilon}^{BSM}}\right)$, i.e., Eqs. (1a) and (1b) are calculated by using Simpson's rule and taking$\Delta\alpha = 0.1$.

Step 10. We conduct a sensitivity analysis of the accuracy of binomial up-and-down models in approximating BSM prices. We then identify the best fit to the BSM model. This sensitivity analysis is performed sequentially, using three criteria to differentiate the options:

(a) the moneyness grade of the option (in the money, at the money or out of the money),
(b) the type of option (call or put) and
(c) volatility scenario.

In this regard, we consider three types of volatility: low (*ε*=0.01, 0.05, 0.25), medium (*ε*=0.4, 0.5 and 0.6) and high volatility (*ε*=0.75, 0.95, 0.99).

In all of the sets of options defined by every criterion, we evaluate the goodness of each fuzzy binomial approach independently for each periodicity *h* but also by grouping *h* into three different frequencies: low frequency (*h* = annual, semiannual or quarterly), middle frequency (*h* = monthly or weekly) and high frequency (*h* = daily and half daily).

For a set of options owing to the segmentation of a given criterion (for example, the group of "in the money" options if the criterion is moneyness degree) and a periodicity/group of periods (for example, *h*=504), we follow the next steps:

Step 10.1. For each BSM price, among the three binomial specifications, CRR, RB and TRIG provide the best approximations.

Step 10.2. State for the whole set of evaluated options (for example, in the money options) and for a concrete *h*, the number of "wins" by each method and the proportion that these wins suppose.

Step 10.3. If the goodness of fit of the three binomial methods for approximating the BSM is similar, the proportion of wins of any method must not be significantly different at 1/3. Therefore, if we symbolize the proportion of wins of the *i*th method as $\pi_i > 1/3$, we perform a test on proportions for the method with the highest rate of wins. Assuming that the *i*th method is allegedly the best method, we test the null hypothesis $p_i = 1/3$, where p_i is the true proportion of wins according to the *i*th binomial specification. The statistic used to perform the assessment is as follows:

$$z = \frac{\pi_i - 1/3}{\sqrt{\frac{\frac{1}{3}\left(1-\frac{1}{3}\right)}{n}}} \sim N(0,1), \quad (22)$$

where *n* is the number of options in the set in which we are evaluating CRR, RB and TRIG, and where $N(0,1)$ is the standard Gaussian distribution. The acceptance of $p_i = 1/3$ implies that the *i*th method does not provide an approximation of the BSM systematically better than the alternative binomial specifications do.

Note that considering a greater number of percentiles for volatility (ε) and periods (h) increases the power of the hypothesis test conducted on the proportion using (22) for a given significance level, which in our case is 5%.

4.3. Results of the comparative assessment

The results of the empirical comparative assessment shown in this section have been fully implemented with a worksheet. Table 4 displays the sensitivity analysis of the errors of fuzzy binomial models in approximating fuzzy BSM prices in the volatility scenario

$\tilde{\sigma}_{0.5}$, which is presented in Table 3. In this scenario, $\sigma_{0.5_0} = [0.12860,\ 0.17448]$ and $\sigma_{0.5_1} = 0.14780$. We have not displayed the results in other $\tilde{\sigma}_{\varepsilon}\sigma$ because the observed patterns were essentially the same. These errors were calculated with Eqs. (21) and (22). We can check that:

a) In general, the in the money options (calls with *K*=0.9 and puts with *K*=1.1) are better approximated than at-the-money options (*K*=1), and these latter options are better fitted than the out-of-the-money options (*K*=1.1 for call options and *K*=0.9 for put options).
b) Even though all the binomial methodologies converge to the BSM since Eqs. (21) and (22) in $h = 1/504$ are rarely above 0.01%, this convergence is sometimes not monotonic with respect to h since it can be checked that several times the distance to BSM of binomial prices for $h = 1/2$ are greater than those for $h = 1$.

Tables 5a and 5b show the proportions in which each binomial method is superior to the other methods when approximating BSM with different options according to the moneyness degree. That is, we implement sensitivity analysis steps 10.1, 10.2 and 10.3 by using grouping criterion (a). The statistical test in Eq. (22) suggests that in the set of in-the-money options, there is no move modelling better than in the other options. On the other hand, a general pattern is that the RB provides the best approaches to the BSM in the cases of out-of-the-money and at-the-money options.

Tables 6a and 6b display the proportion of wins of every assessed binomial model differentiating in the sensitivity analysis between call and put options. Therefore, we perform steps 10.1, 10.2, and 10.3 by using grouping criterion (b). By using Eq. (22), the RB generally provides a better approach. On the other hand, the poorer approximation comes from TRIG in the case of call options and from CRR in the case of put options.

Tables 7a and 7b show the proportions in which every binomial approach outperforms the alternatives to fit BSMs differentiating in the sensitivity analysis of low-, medium- and high-volatility scenarios. Therefore, we perform steps 10.1, 10.2, and 10.3 by using the grouping criterion (c). By using Eq. (22), we can make the general statement that RB provides the closest prices to BSM.

Table 4. Evolution of $error^{Bin_i}_{K,\varepsilon,h}$ by fuzzy CRR, RB and TRIG in volatility scenario $\tilde{\sigma}_{0.5}$

	Call options (*K*=0.9)			Put options (*K*=0.9)		
h	CRR	RB	TRIG	CRR	RB	TRIG
1	0.493%	0.898%	0.683%	2.933%	5.344%	2.865%
1/2	1.030%	0.980%	1.239%	6.131%	5.829%	5.881%
1/4	0.377%	0.495%	0.490%	2.244%	2.945%	2.119%
1/12	0.087%	0.098%	0.088%	0.517%	0.583%	0.524%
1/48	0.026%	0.028%	0.034%	0.158%	0.168%	0.149%
1/252	0.005%	0.006%	0.006%	0.032%	0.033%	0.032%
1/504	0.003%	0.003%	0.003%	0.016%	0.017%	0.016%
	Call options (*K*=1)			Put options (*K*=1)		
h	CRR	RB	TRIG	CRR	RB	TRIG
1	6.309%	6.129%	6.873%	6.309%	6.129%	5.992%
1/2	2.847%	1.732%	2.548%	2.847%	1.732%	3.054%
1/4	1.502%	0.686%	1.340%	1.502%	0.686%	1.613%
1/12	0.515%	0.058%	0.458%	0.515%	0.058%	0.555%
1/48	0.130%	0.069%	0.115%	0.130%	0.069%	0.140%
1/252	0.025%	0.022%	0.022%	0.025%	0.022%	0.027%
1/504	0.012%	0.005%	0.011%	0.012%	0.005%	0.013%
	Call options (*K*=1.1)			Put options (*K*=1.1)		
h	CRR	RB	TRIG	CRR	RB	TRIG
1	3.724%	2.009%	4.369%	0.744%	0.401%	0.515%
1/2	5.479%	5.560%	5.929%	1.095%	1.111%	0.944%
1/4	2.630%	1.842%	2.863%	0.526%	0.368%	0.442%
1/12	0.591%	0.369%	0.543%	0.118%	0.074%	0.141%
1/48	0.172%	0.154%	0.169%	0.034%	0.031%	0.035%
1/252	0.033%	0.029%	0.035%	0.007%	0.006%	0.006%
1/504	0.016%	0.015%	0.017%	0.003%	0.003%	0.003%

Table 5a. Proportions in which CRR, RB, and TRIG provide the best fit to BSM depending on the options moneyness degree for *h*=1, 1/2, 1/4, 1/12, 1/48, 1/252, 1/504

In the money options	At the money options	Out of the money options

h	CRR	RB	TRIG	CRR	RB	TRIG	CRR	RB	TRIG
1	38.89	38.89	22.22	0.00	55.56**	44.44	5.56	50.00	44.44
1/2	38.89	27.78	33.33	0.00	66.67***	33.33	33.33	38.89	27.78
1/4	33.33	33.33	33.33	0.00	100.00***	0.00	16.67	50.00	33.33
1/12	11.11	50.00	38.89	0.00	100.00***	0.00	38.89	50.00	11.11
1/48	38.89	33.33	27.78	0.00	100.00***	0.00	22.22	38.89	38.89
1/252	27.78	50.00	22.22	0.00	94.44***	5.56	16.67	61.11***	22.22
1/504	33.33	38.89	22.22	0.00	94.44***	5.56	16.67	50.00	27.78

Notes: (1) Proportions are percentages. (2) “*”, “**”, and “***” represent significance at the 10%, 5% and 1% levels, respectively.

Table 5b. Proportions in which CRR, RB, and TRIG provide the best fit to BSM depending on the options moneyness degree and grouping *h* into low-, medium- and high-frequency sets

	In the money options			At the money options			Out of the money options		
	CRR	RB	TRIG	CRR	RB	TRIG	CRR	RB	TRIG
Low frequency	37.04	33.33	29.63	0.00	74.07***	25.93	18.52	46.30**	35.19
Medium frequency	25.00	41.67	33.33	0.00	100.00***	0.00	30.56	44.44	25.00
High frequency	30.56	44.44	22.22	0.00	94.44***	5.56	16.67	55.56**	25.00

Notes: (1) Proportions are percentages. (2) “*”, “**”, and “***” represent significance at the 10%, 5% and 1% levels, respectively.

Table 6a. Proportions in which CRR, RB, and TRIG provide the best fit to BSM, distinguishing between call and put options for *h*=1, 1/2, 1/4, 1/12, 1/48, 1/252, and 1/504.

	Call options			Put options		
h	CRR	RB	TRIG	CRR	RB	TRIG
1	19.23	57.69***	23.08	11.54	42.31	46.15
1/2	42.31	57.69***	0.00	7.69	50.00*	42.31
1/4	26.92	57.69***	15.38	7.69	61.54**	30.77
1/12	23.08	61.54***	15.38	11.54	65.38***	23.08
1/48	26.92	53.85**	19.23	15.38	53.85**	30.77
1/252	23.08	65.38***	11.54	7.69	61.54***	30.77
1/504	34.62	57.69***	7.69	0.00	53.85**	34.62

Notes: (1) Proportions are percentages. (2) “*”, “**”, and “***” represent significance at the 10%, 5% and 1% levels, respectively.

Table 6b. Proportions in which CRR, RB, and TRIG provide the best fit to BSM, distinguishing between call and put options and grouping *h* into low-, medium- and high-frequency sets

	Call options			Put options		
	CRR	RB	TRIG	CRR	RB	TRIG
Low frequency	29.49	57.69***	12.82	8.97	51.28***	39.74
Medium frequency	25.00	57.69***	17.31	13.46	59.62***	26.92
High frequency	28.85	61.54***	9.62	3.85	57.69***	32.69

Notes: (1) Proportions are percentages. (2) “*”, “**”, and “***” represent significance at the 10%, 5% and 1% levels, respectively.

Table 7a. Proportions in which CRR, RB, and TRIG provide the best fit to BSM for differentiating three volatility scenarios with *h*=1, 1/2, 1/4, 1/12, 1/48, 1/252, and 1/504.

	Low volatility scenarios			Medium volatility scenarios			High volatility scenarios		
h	CRR	RB	TRIG	CRR	RB	TRIG	CRR	RB	TRIG
1	16.67	38.89	44.44	16.67	38.89	44.44	11.11	72.22***	16.67
1/2	16.67	61.11**	22.22	16.67	61.11***	22.22	38.89	44.44	16.67
1/4	16.67	55.56	27.78	16.67	66.67***	16.67	16.67	61.11***	22.22
1/12	22.22	50.00	27.78	16.67	72.22***	11.11	11.11	72.22***	16.67
1/48	11.11	77.78***	11.11	22.22	55.56**	22.22	27.78	33.33	38.89
1/252	16.67	66.67**	16.67	11.11	72.22***	16.67	16.67	55.56**	27.78
1/504	5.56	77.78***	16.67	16.67	50.00	16.67	27.78	38.89	33.33

Notes: (1) Proportions are percentages. (2) “*”, “**”, and “***” represent significance at the 10%, 5% and 1% levels, respectively.

Table 7b. Proportions in which CRR, RB, and TRIG provide the best fit to BSM for differentiating three volatility scenarios and grouping *h* into low-, medium- and high-frequency sets

	Low volatility scenarios			Medium volatility scenarios			High volatility scenarios		
	CRR	RB	TRIG	CRR	RB	TRIG	CRR	RB	TRIG
Low frequency	16.67	51.85***	31.48	16.67	55.56***	27.78	22.22	59.26***	18.52

Medium frequency	19.44	52.78***	27.78	16.67	69.44***	13.89	13.89	66.67***	19.44
High frequency	13.89	72.22***	13.89	16.67	63.89***	19.44	22.22	44.44*	33.33

Notes: (1) Proportions are percentages. (2) "*", "**", and "***" represent significance at the 10%, 5% and 1% levels, respectively.

5. Discussion

The approach of Cox, Ross and Rubinstein [15] has been used unanimously in FBOP to explicitly link up and down moves and their probabilities with the volatility of subjacent assets. However, the binomial option pricing literature has provided several models [12] to parameterize binomial multipliers and probabilities; thus, there is no reason to avoid these alternative parameterizations in fuzzy binomial modelling [44]. Given that one of the main justifications for using binomial lattices where there is not a closed formula for option prices is their ability to converge toward BSM prices [18], it seems logical to consider the method used to parameterize its multipliers and neutral-risk probabilities. This reflection has led us to compare the capacity to approximate the fuzzy BSM of the CRR [15] with the parameterizations by Rendleman and Bartter [31] (RB) and by Trigeorgis [34] (TRIG). This type of convergence analysis, which is common in the option pricing literature [12, 18, 20], is very scarce in the fuzzy option pricing literature [1].

To analyse this issue, we developed an empirical application by using data from the Spanish derivative market, specifically, the daily average values of the IBEX-35 futures index during the period from January 2013 to January 2023. The options assessed in this paper were priced by using historical fuzzy volatilities fitted by using approach [32]. They were obtained by superposing statistical confidence intervals for the variance of logarithmic differences in the daily values of the index.

Tables 5a and 5b show the rates at which every binomial parameterization is the best according to the degree of moneyness of the options. We observe.

a) There is no parameterization that is significantly better than the others in the set of "in the money" options.

b) RB provides better approaches to BSM in the case of out-of-the-money options and, above all, in options with null intrinsic values. This issue is especially clear in Table 5b, where we can check that the percentages of wins RB for low-frequency *h* (46.30%, $p<0.05$) and high-frequency *h* (55.56%, $p<0.05$) are clearly significant.

c) As far as the money options are concerned, the RB provides consistently better approximations to the BSM than TRIG and CRR do, and this greater performance is highly significant. Notably, in terms of "at the money" option prices, the CRR did not reach any best approach.

Tables 6a and 6b display the proportion of wins of every assessed binomial model differentiating between call and put options. The results suggest that RB generally provides a better approach. Therefore, we outline the following:

a) Regarding call options, RB is always the "best" binomial approach to BSM, and this superiority is always significant.
b) Tables 6a and 6b also show that RB is also always the "best" binomial approach to BSM in the case of put options, and often, this best performance is significant. This fact is strongly clear in Table 6b, where it can be observed that the proportion in which RB provides the greatest convergence to BSM oscillates between 58% (in low and medium frequencies) and 61% (in high frequencies), and it can always be rejected that these proportions are 1/3 ($p<0.01$).

Tables 7a and 7b show the rates at which every binomial approach outperforms the alternatives for fitting BSM for differentiating among the three volatility scenarios. Generally, RB provides the closest prices to BSM. So:

a) With respect to low-volatility scenarios, Table 7a shows that RB has the greatest proportion in all *h*, with the exception of h=1. Table 7b shows that if we consider only low, medium and high frequencies, RB always attains the greatest win rates, which oscillate between 52% and 72% ($p<0.01$).
b) When analysing the results in medium volatility scenarios, Table 7a shows that RB again provides a better approach to BSM prices in practically all *h*, and this better approximation is often significant. Table 7b shows that in all the sets of frequencies, RB significantly provides the closest prices to BSM.
c) In the scenarios with the highest volatility, RB also provides, with the exception of weekly *h*, the most accurate approximation to BSM, and this superiority is often significant. Table 7b shows that when we group the frequencies into low, medium and high, the better performance of RB approach is strongly evident.

The results obtained under a fuzzy environment in this paper are consistent with those of [12,20], which demonstrate that the assessed binomial modelling approaches converge to the BSM values. They also show, similar to [12,20], that not all binomial models converge at the same rate, meaning that the optimal structure of up and down moves depends on the option pricing setting and the closed-form formula being analysed.

In our study, although the best-performing model is often RB, this is not the case in all the scenarios. There are instances where the best approximation to the BSM is provided by TRIG or CRR. This finding has significant implications for fuzzy–random option pricing via binomial lattices. The mainstream approach in FBOP typically models binomial moves with CRR without considering alternative formulations [3,4,17,29,39-45], even though the literature [12] has proposed over 10 different models that may be more appropriate. This study provides an analytical framework that can assist in selecting the optimal binomial move modelling approach in a fuzzy setting.

6. Conclusions and further research

The contribution of this paper is twofold. On the one hand, it has provided an overview of fuzzy random option pricing developments centred on papers focused on the fuzzy binomial approach (FBOP), which is the mainstream of fuzzy-random option pricing in discrete time. It must be emphasized that the use of binomial models is very popular because of the confluence of two nice properties: they are flexible enough to model many optionality forms, and moreover, by adequately linking up and down factors with the volatility of the subjacent asset, they provide similar prices to the BSM formula [6,23]. Therefore, binomial option pricing allows the application of BSM in contexts where that formula cannot be applied directly.

Our empirical applications show that, as we expected, the assessed methods (CRR, RB and TRIG) converge to BSM prices when the frequency of price movements tends to infinity. However, as a general setting, we can state that RB parameterization provides prices closer to those of BSM than those of CRR and TRIG when that frequency is finite. This statement depends neither on the periodicity used to apply the fuzzy binomial method nor on the degree of market volatility. It also does not depend on whether we are pricing calls or put options. In fact, among the evaluated parameterizations, CRR provides the worst approximations to BSM prices for “at the money” options and for the put options. These findings underscore the importance of carefully selecting the parameterization of the up and down factors, as well as the risk-neutral probabilities, in future studies using a fuzzy binomial approach to option pricing, as achieving closer convergence to the BSM is a highly desirable characteristic in discrete option pricing models.

All papers on fuzzy option pricing with binomial lattices introduce uncertainty on up- and down moves by using fuzzy numbers, often with a linear shape. Consequently, risk-neutral probabilities also become fuzzy numbers. An extension of the present study could involve replicating it using fuzzy volatilities modelled with triangular or generalized triangular fuzzy numbers. Additionally, future comparative analyses in a fuzzy environment could explore further alternatives for modelling binomial moves beyond those presented in [15, 31, 34] and those outlined in [12].

The options assessed in this paper were priced by using historical fuzzy volatilities. Fuzzy implicit volatility methods such as those based on fuzzy regression [28] could be reliable alternatives.

More complex forms of uncertain quantities, such as intuitionistic fuzzy numbers [1], have rarely been used in FBOP and even others, such as picture fuzzy sets [40], never. Thus, a natural extension of FBOP is the use of this kind of instrument to model uncertainty over the parameters that govern stochastic movements of the prices of subjacent assets. However, the introduction of these tools to model uncertainty sources such as vagueness, ambiguity or incompleteness implies that the calculations may become less parsimonious and that the input data may require the estimation of more parameters; thus, it may be more difficult to apply analytical results in practice.

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