



# Irreducible minimum cost spanning tree problems with revenues

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## Abstract

The current paper analyzes minimum cost spanning tree problems having irreducible costs while incorporating revenues. We prove that, in this context, the core of the associated cost-revenues game (the  $r$ -core) is non-empty. In particular, we find an stable allocation in the  $r$ -core, based on the CEL bankruptcy rule, that ensures fairness by distributing revenues only among agents that belong to every effective coalition. Therefore, our findings contribute to the literature by identifying structural conditions that guarantee stability in minimum cost spanning tree problems with revenues, overcoming previous results which showed that the  $r$ -core could be empty.

**Keywords** Minimum cost spanning tree problem · Irreducible · Revenue · Core · Effective coalition

**Mathematics Subject Classification** 91B06 · 91A35

**JEL Classification** C71 · D63 · D71

## 1 Introduction

The minimum cost spanning tree (*mcst*) problem is a key topic in combinatorial optimization and cooperative game theory. It concerns the efficient connection of a set of agents to a central source while minimizing costs. This problem has broad applications in network design, logistics, and infrastructure planning. Traditional *mcst* models primarily focus on cost allocation among participating agents. However, the introduction of revenues brings new strategic challenges, requiring a re-evaluation of stability and fairness within cooperative structures.

Estévez-Fernández and Reijnierse (2014) examine the *mcst* problem in the context of revenues generated by the provided service. This leads to a crucial question:

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how should the net revenues from the cooperation be distributed among the agents? Considering revenues adds an extra layer of complexity, particularly regarding the core of the associated cooperative game—the set of allocations where no coalition has an incentive to deviate (which we refer to as the  $r$ -core, where the letter  $r$  stands for revenues). Estévez-Fernández and Reijnierse (2014) show that the  $r$ -core may be empty, even in elementary problems in which connection costs take only two distinct values: low and high cost.

A defining characteristic of  $mcst$  problems with revenues is that certain coalitions may increase their benefits by excluding some members from accessing the revenue-generating service. As a result, some individuals may remain disconnected in the optimal cost-revenue spanning tree. Estévez-Fernández and Reijnierse (2014) and Subiza et al. (2025b) establish conditions ensuring the non-emptiness of the  $r$ -core in elementary  $mcst$  problems. These conditions are based on the behavior of effective coalitions, defined as coalitions that maximize potential profits. Furthermore, Subiza et al. (2025b) provide a necessary condition for the  $r$ -core to be non-empty: at least one agent must be included in every effective coalition. This result applies to general  $mcst$  problems with revenues. Despite these advances, the sufficient conditions identified thus far do not encompass all possible scenarios that guarantee a non-empty  $r$ -core. Further research is needed to fully characterize the conditions under which stability is ensured in  $mcst$  problems with revenues.<sup>1</sup>

The notion of irreducible cost matrix was initially introduced in Bird (1976). An irreducible cost matrix entails that the costs are tight and fully justify the structure of the minimum spanning tree: lowering the cost of any non-tree edge could alter the optimal solution. In Bergantiños and Vidal-Puga (2007), this concept is the key for defining the Folk solution (independently proposed by Feltkamp et al. 1994), which is obtained as the Shapley value of the stand-alone game associated with the irreducible cost matrix. Furthermore, several additional properties emerge when working with irreducible cost matrices (see, for instance, Bogomolnaia and Moulin 2010; Bergantiños and Vidal-Puga 2021):

- The common and private stand-alone games coincide and this game is concave.
- The Kar solution (Kar 2002) coincides with the Folk solution, so it belongs to the core, contrary to what happens under other cost configurations.
- The Folk solution in elementary  $mcst$  problems can be easily computed (Bogomolnaia and Moulin 2010; Subiza et al. 2016).

The introduction of revenues in irreducible  $mcst$  problems adds an additional feature. The fundamental challenge lies in ensuring that no subgroup of agents has an incentive to deviate when revenues are distributed, which could potentially lead to fragmentation even in an irreducible structure. The main contribution of this paper is the analysis of the non-emptiness of the  $r$ -core in irreducible  $mcst$  games with revenues.

The remainder of this paper is organized as follows. Section 2 introduces the formal definitions of the  $mcst$  problem with revenues. Section 3 analyzes irreducible and elementary  $mcst$  games, establishing their structural properties and analyzing the

<sup>1</sup> To overcome the possible emptiness of the  $r$ -core, Subiza et al. (2025a) suggest the use of alternative notions of stability in these cases.

non-emptiness of the  $r$ -core. Section 4 generalizes the results in the previous section for general irreducible  $mcst$  games with revenues. Some final comments close the paper.

## 2 Preliminaries

### 2.1 Minimum cost spanning tree problem with revenues

A minimum cost spanning tree problem consists of a set of agents  $N = \{1, 2, \dots, n\}$ ,  $n \geq 2$ , that want to connect (directly, or throughout other agents) to a source node  $\omega$  minimizing the total connection cost. The set of nodes is denoted by  $N_\omega = N \cup \{\omega\}$ . An edge  $(i, j) \in N_\omega \times N_\omega$  is a potential direct link between nodes  $i$  and  $j$  with associated cost  $c_{ij} \geq 0$ . As usual, we assume that:

- (a)  $c_{ii} = 0$ , for all  $i \in N_\omega$ ; and
- (b)  $c_{ij} = c_{ji}$ , for all  $i, j \in N_\omega$ .

The  $mcst$  problem will be denoted by  $(N_\omega, \mathbf{C})$ , where  $\mathbf{C}$  represents the symmetric cost matrix.

A path connecting two nodes  $i, j \in N_\omega$  is denoted by  $p_{ij}$  and consists of a sequence of edges that join an ordered set of  $m \geq 2$  different nodes, represented as  $\{i = h_1, h_2, \dots, h_m = j\}$ . Formally, the path is given by  $p_{ij} = \{(h_r, h_{r+1}) : r = 1, 2, \dots, m - 1\}$ , where each edge  $(h_r, h_{r+1})$  represents a direct connection between consecutive nodes in the sequence.

A minimum cost spanning tree is a structure connecting all agents to the source at minimum cost. In an  $mcst$  problem with revenues, each agent  $i \in N$  obtains a revenue  $b_i \geq 0$  from being connected to the source  $\omega$ . The net revenue of a coalition  $S \subseteq N$  is given by

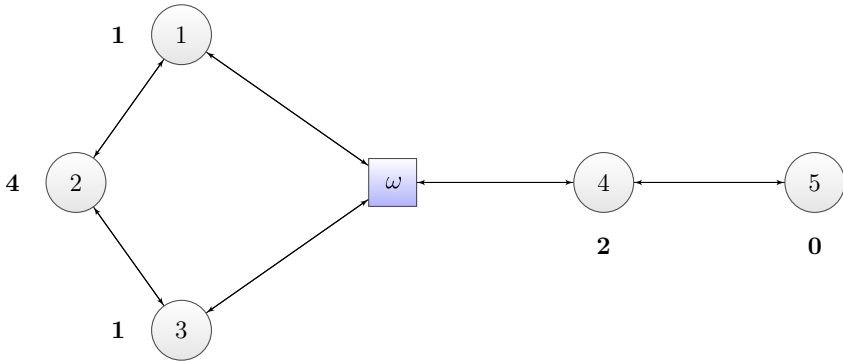
$$\pi(S) = \sum_{i \in S} b_i - c(S),$$

where  $c(S)$  represents the minimum cost required to connect all members of  $S$  to  $\omega$ . In Estévez-Fernández and Reijnders (2014), the model is formulated under a private property approach: agents in a coalition  $S$  cannot use nodes outside  $S$  when connecting to the source. In this paper, we adopt a common property formulation, so that coalitions may route through nodes outside  $S$ . Note that, since we work with irreducible  $mcst$  problems, both approaches lead to the same revenue allocation in our setting. This modeling choice therefore does not affect the validity or generality of our results.

### 2.2 The $r$ -core

A transferable-utility (TU) game is a pair  $(N, v)$ , where  $N$  is a finite set of agents and  $v : 2^N \rightarrow \mathbb{R}$  is the characteristic function. The characteristic function of the cooperative TU game induced by the  $mcst$  problem with revenues is

$$v(S) = \max\{\pi(R) : R \subseteq S\}.$$



**Fig. 1** Five-agent *mcst* problem with revenues. The cost of each link appearing in the picture is  $c_{ij} = 1$ , whereas the cost of the undrawn links is  $c_{ij} = 5$ . The boldfaced numbers at the nodes represent the revenues that the corresponding agents obtain

Then, the pair  $(N, v)$  denotes the cooperative *mcst* game with revenues. When agents agree to cooperate, they must determine how to share the net revenue they generate. That is, to decide the amount  $x_i$  to be allocated to agent  $i$ , such that  $\sum_{i \in N} x_i = v(N)$ . The stability of allocations is analyzed through the *r*-core.

**Definition 1** The core of a TU game  $(N, v)$  is the set

$$\text{Core}(N, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \right\}.$$

The *r*-**core** is the core of an *mcst* game with revenues  $(N, v)$ .

The following example illustrates the construction of the characteristic function of a five-agent game with revenues.

**Example 1** Consider the *mcst* problem with revenues represented in Fig. 1.

The construction of the cost-revenues game is illustrated in Table 1.

It is immediate to observe that the following allocation belongs to the *r*-core

$$x_1^* = 0, \quad x_2^* = 3, \quad x_3^* = 0, \quad x_4^* = 1, \quad x_5^* = 0.$$

### 3 Irreducible and elementary *mcst* problems

Elementary *mcst* problems play a central role in our analysis. By restricting edge costs to only two distinct values,  $\kappa_1 < \kappa_2$ , they capture the essential tension between “cheap” and “expensive” connections while keeping the structure analytically tractable. This simplification is not merely technical: it highlights the fundamental mechanisms behind coalition formation and cost allocation, and provides a natural benchmark for understanding more general *mcst* problems with revenues.

**Table 1** Coalitions  $S$ , their costs  $c(S)$ , revenues  $b(S) = \sum_{i \in S} b_i$ , net revenues  $\pi(S)$ , and values  $v(S)$  for the  $mcst$  problem in Fig. 1

$S$	$c(S)$	$b(S)$	$\pi(S)$	$v(S)$
{1}	1	1	0	0
{2}	2	4	2	2
{3}	1	1	0	0
{4}	1	2	1	1
{5}	2	0	-2	0
{1, 2}	2	5	3	3
{1, 3}	2	2	0	0
{1, 4}	2	3	1	1
{1, 5}	3	1	-2	0
{2, 3}	2	5	3	3
{2, 4}	3	6	3	3
{2, 5}	4	4	0	2
{3, 4}	2	3	1	1
{3, 5}	3	1	-2	0
{4, 5}	2	2	0	1
{1, 2, 3}	3	6	3	3
{1, 2, 4}	3	7	4	4
{1, 2, 5}	4	5	1	3
{1, 3, 4}	3	4	1	1
{1, 3, 5}	4	2	-2	0
{1, 4, 5}	3	3	0	1
{2, 3, 4}	3	7	4	4
{2, 3, 5}	4	5	1	3
{2, 4, 5}	4	6	2	3
{3, 4, 5}	3	3	0	1
{1, 2, 3, 4}	4	8	4	4
{1, 2, 3, 5}	5	6	1	3
{1, 2, 4, 5}	4	7	3	4
{1, 3, 4, 5}	4	4	0	1
{2, 3, 4, 5}	4	7	3	4
$N$	5	8	3	4

**Definition 2** An **elementary  $mcst$  problem**, also known as  $2\text{-}mcst$  problem, is one in which each edge has only two possible costs, low and high cost: for all  $i, j \in N_\omega$ , with  $i \neq j, c_{ij} \in \{\kappa_1, \kappa_2\}$ , with  $0 \leq \kappa_1 < \kappa_2$ .

On the other hand, irreducible  $mcst$  problems constitute the setting of our study. An irreducible cost matrix ensures that the edge costs exactly support the structure of the minimum spanning tree: any reduction in the cost of a non-tree edge may change the optimal solution.

**Definition 3** Given an *mcst* problem  $(N_\omega, \mathbf{C})$  the **irreducible** cost matrix  $\mathbf{C}^*$  is defined by

$$c_{ij}^* = \min_{p_{ij}} \max \{c_e : e \in p_{ij}\},$$

where the minimum bears on all paths  $p_{ij}$  connecting nodes  $i$  and  $j$ . Thus,  $c_{ij}^*$  is the smallest number  $z$ , such that any path from  $i$  to  $j$  contains at least one edge with cost at least  $z$ . An *mcst* problem  $(N_\omega, \mathbf{C})$  is **irreducible** if  $\mathbf{C}^* = \mathbf{C}$ .

The notion of an ultrametric is related to the irreducibility of the cost matrix.

**Definition 4** (Ando and Kato 2010) Given a finite set  $Q$ , a function  $\varpi : Q \times Q \rightarrow \mathbb{R}$  is an **ultrametric** if, for every distinct  $i, j, k \in Q$ , the following inequality holds:

$$\varpi(i, j) \leq \max \{\varpi(i, k), \varpi(k, j)\}.$$

Equivalently,  $\varpi$  is an ultrametric if and only if, for every distinct  $i, j, k \in Q$ , the maximum value among  $\varpi(i, j)$ ,  $\varpi(j, k)$ , and  $\varpi(i, k)$  is attained by at least two of these pairs. An *mcst* problem  $(N_\omega, \mathbf{C})$  is said to be **ultrametric** if the cost matrix  $\mathbf{C}$  is an ultrametric, i.e., for every distinct  $i, j, k \in N_\omega$ ,  $c_{ij} \leq \max \{c_{ik}, c_{kj}\}$ .

The following results highlight the close connection between irreducible *mcst* problems and ultrametric cost matrices.

**Lemma 1** Every irreducible *mcst* problem is ultrametric.

**Proof** Let  $(N_\omega, \mathbf{C})$  be an irreducible *mcst* problem and  $i, j \in N_\omega$ . By definition, we have  $\mathbf{C}^* = \mathbf{C}$ , which implies that

$$c_{ij} = c_{ij}^* = \min_{p_{ij}} \max \{c_e : e \in p_{ij}\}.$$

For any  $k \neq i$  and  $k \neq j$ , consider the path  $p_{ij} = \{(i, k), (k, j)\}$  connecting nodes  $i$  and  $j$ . By construction, it follows that:

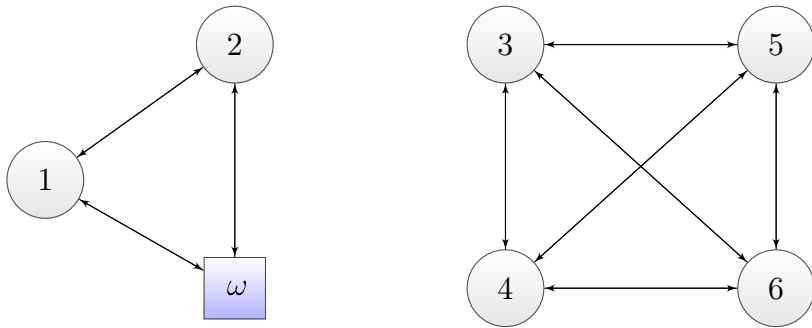
$$c_{ij} = c_{ij}^* \leq \max \{c_{ik}, c_{kj}\}.$$

Since this inequality holds for all  $i, j, k \in N_\omega$ , the cost function  $\mathbf{C}$  satisfies the ultrametric property.  $\square$

**Lemma 2** Every ultrametric elementary *mcst* problem is irreducible.

**Proof** Let  $(N_\omega, \mathbf{C})$  be an *mcst* problem, such that  $\mathbf{C}$  is ultrametric and elementary. Let  $i, j \in N_\omega$  with  $i \neq j$ . Consider the following two cases:

(a) If  $\max \{c_e : e \in p_{ij}\} = \kappa_2$  for all paths  $p_{ij}$ , then by definition,  $c_{ij}^* = \kappa_2$ . Since a valid path is the direct connection  $p_{ij} = \{(i, j)\}$ , it follows that  $c_{ij} = \kappa_2$ .



**Fig. 2** Irreducible and elementary *mcst* problem.  $N = \{1, 2, 3, 4, 5, 6\}$ . The cost of each edge appearing in the picture is  $c_{ij} = \kappa_1$ , whereas the cost of the undrawn edges is  $c_{ij} = \kappa_2$

- (b) If  $\max \{c_e : e \in p_{ij}\} = \kappa_1$  for some path  $p_{ij}$ , then  $c_{ij}^* = \kappa_1$ , and there exists a path  $p_{ij}$  with nodes  $\{i = h_1, h_2, \dots, h_m = j\}$ , such that  $c_{e_r} = \kappa_1$  for every edge  $e_r = (h_r, h_{r+1})$ , where  $r = 1, 2, \dots, m - 1$ .

Since the problem is ultrametric, this property ensures that if  $c_{e_1} = \kappa_1$  and  $c_{e_2} = \kappa_1$ , then the direct edge cost  $c_{ih_2}$  must also be  $\kappa_1$ . Repeating this argument along the path, we conclude that  $c_{ij} = \kappa_1$ .

Thus, in all cases, we have  $c_{ij}^* = c_{ij}$ , proving that the problem is irreducible. □

As a consequence, for elementary *mcst* problems, being irreducible and ultrametric are equivalent conditions.

**Corollary 1** *An elementary mcst problem is irreducible if and only if is ultrametric.*

### 3.1 Characterization of irreducible and elementary *mcst* problems

In irreducible and elementary *mcst* problems, the structure of the *mcst* exhibits key properties that distinguish it from general *mcst* formulations. Since these problems involve only two possible edge costs,  $\kappa_1$  (low) and  $\kappa_2$  (high), the resulting *mcst* follows a hierarchical organization where low-cost edges form the backbone of the network. In an elementary *mcst* problem, every agent is connected to the source either directly or indirectly through a sequence of low-cost edges whenever possible. In an irreducible setting, the cost function satisfies the ultrametric property, ensuring that the spanning tree maintains a block structure in which coalitions are formed based on their relative costs. The intuition behind this structure is that elements within the same block are connected with the lower cost  $\kappa_1$ , ensuring minimum internal connection costs. In contrast, elements from different blocks incur the higher cost  $\kappa_2$  to be connected among them. This structure is illustrated by Fig. 2, and it generalizes to larger matrices by recursively partitioning indices into nested clusters.

Our next result characterizes irreducible and elementary *mcst* problems.

**Theorem 1** *An  $mcst$  problem  $(N_\omega, \mathbf{C})$  is irreducible and elementary if and only if there exists a partition of  $N_\omega$  into  $t + 1$  components,  $H^\rho$  for  $\rho = 0, 1, 2, \dots, t$ , with  $\omega \in H^0$ , such that:*

- (a) *For all  $i, j \in H^\rho$ , with  $i \neq j$ ,  $c_{ij} = \kappa_1$ .*
- (b) *For all  $i \in H^\rho, j \in H^{\rho'}$ , with  $\rho \neq \rho'$ ,  $c_{ij} = \kappa_2$ .*

**Proof** The existence of a partition of  $N_\omega$  into  $H^\rho$  for  $\rho = 0, 1, 2, \dots, t$ , satisfying conditions (a) and (b), ensures that the cost matrix  $\mathbf{C}$  takes only two distinct values and satisfies the ultrametric property. Consequently, by Corollary 1, the  $mcst$  problem is elementary and irreducible.

Conversely, suppose that the  $mcst$  problem  $(N_\omega, \mathbf{C})$  is irreducible and elementary. We define the following binary relation on  $N_\omega$ :

$$i, j \in N_\omega, \quad i \sim j \iff i = j \text{ or } c_{ij} = \kappa_1.$$

Since  $\mathbf{C}$  is irreducible, this is an equivalence relation. Reflexivity and symmetry are immediate, while transitivity follows directly from the ultrametric property of  $\mathbf{C}$ .

Let  $H^\rho$  denote the equivalence classes in the corresponding quotient set. By construction, these classes satisfy condition (a). Condition (b) follows from the elementary cost property, which guarantees that connections between different equivalence classes have cost  $\kappa_2$ . □

Note that  $H^0$  may consist solely of the source, but if it includes other elements, then  $c_{i\omega} = \kappa_1$  for all  $i \in H^0$ , with  $i \neq \omega$ .

Theorem 1 characterizes the structural conditions under which an  $mcst$  problem is irreducible and elementary. This result provides a fundamental framework for analyzing irreducible  $mcst$  games with revenues.

### 3.2 The $r$ -core of irreducible and elementary $mcst$ games with revenues

When an  $mcst$  problem  $(N, \mathbf{C})$  is elementary or irreducible its associated TU game  $(N, v)$  is said to be an elementary  $mcst$  game or an irreducible  $mcst$  game, respectively.

#### 3.2.1 Effective coalitions and the $r$ -core

The  $r$ -core ensures stability by preventing any coalition from being allocated less than its collective worth. In this context, not all coalitions contribute equally to the overall profitability of the network. Some subsets of agents are particularly relevant, since they are able to obtain the maximum net revenue that the entire group can obtain. These coalitions are referred to as effective coalitions.

**Definition 5** Given an  $mcst$  game with revenues  $(N, v)$ , a coalition  $E \subseteq N$  is called **effective** if it maximizes the net revenue, i.e.,  $v(N) = \pi(E)$ . A coalition  $E^* \subseteq N$  is called a **minimum effective coalition** if it is effective and moreover,  $E^* \subseteq E$  for any other effective coalition  $E$ .

Intuitively, a minimum effective coalition is the smallest group of agents strictly required to generate the maximum net revenue. Including additional agents does not increase the payoff (and may even reduce it), while excluding any member lowers the benefit. As we shall see, such a coalition does not always exist; but whenever it does, it represents the optimal coalition to be formed to solve the game. On the other hand, effective coalitions play a central role in the discussion about the non-emptiness of the  $r$ -core. The following result presents a necessary condition involving these coalitions.<sup>2</sup>

**Theorem 2** (Subiza et al. 2025b) *If an  $mcst$  game with revenues has a non-empty  $r$ -core, then there is at least an agent  $i$  that belongs to every effective coalition.*

We first analyze the existence of minimum effective coalitions in this framework. It is clear the existence of effective coalitions in any  $mcst$  game with revenues. Nevertheless, the existence of minimum effective coalitions is not ensured, even in the case of non-empty  $r$ -core, as Example 2 shows. In case it exists, by its definition, the minimum effective coalition is unique in each  $mcst$  game with revenues.

**Example 2** Consider the  $mcst$  problem with revenues represented in Fig. 1. Then, the effective coalitions are

$$E_1 = \{1, 2, 4\}, E_2 = \{2, 3, 4\}, E_3 = \{1, 2, 3, 4\}.$$

Observe that there is not a minimum effective coalition. Note that the  $mcst$  problem is elementary but not irreducible.

Contrary to the general case, we can ensure the existence of a minimum effective coalition when the problem is elementary and irreducible. The following result proves that agents that do not belong to some effective coalition do not receive benefits from cooperation.

**Theorem 3** (Subiza et al. 2025b) *For any  $mcst$  game with revenues  $(N, v)$  with non-empty  $r$ -core, if  $x \in \text{Core}(N, v)$  and  $i \notin E$  for some effective coalition  $E$ , then  $x_i = 0$ .*

Consequently, a positive share of the net revenue can be obtained only by the agents that belong to all effective coalitions.

**Lemma 3** *If  $(N, v)$  is an irreducible and elementary  $mcst$  game with revenues, there exists a minimum effective coalition  $E^*$ , such that for any allocation in the  $r$ -core,  $x \in \text{Core}(N, v)$ ,  $x_i = 0$ , for all  $i \in N \setminus E^*$ .*

**Proof** Let  $H^0, H^1, \dots, H^t$  be the partition obtained in Theorem 1, and let  $b$  denote the vector of revenues. Note that this partition is determined independently of the revenues.

Let us define the following coalitions:

$$H_*^0 = \left\{ i \in H^0 \setminus \{\omega\} : b_i > \kappa_1 \right\}$$

<sup>2</sup> The result is an analogous of a well-known property in simple games, which ensures that the core is non-empty if and only if there is at least one veto player (see, for instance, Owen 1982).

and, for  $\rho = 1, 2, \dots, t$ ,

$$H_*^\rho = \{i \in H^\rho : b_i > \kappa_1\}.$$

For each  $\rho \neq 0$ , consider the aggregated remaining benefit for the agents on  $H_*^\rho$  once they connect among themselves inside their component

$$B_\rho = \sum_{i \in H_*^\rho} b_i - \kappa_1(|H_*^\rho| - 1)$$

and, then, define

$$E^* = H_*^0 \cup \bigcup_{\rho \neq 0, B_\rho > \kappa_2} H_*^\rho. \tag{1}$$

From this construction, it is clear that  $E^*$  is effective, since connecting any additional agent does not increase the net revenue. Furthermore, if we remove some agent  $i$  from  $E^*$ , the net revenue decreases in the amount  $b_i - \kappa_1$ , so the new coalition is not effective. Therefore, all effective coalitions need to contain  $E^*$ , which implies that  $E^*$  is a minimum effective coalition.

Finally, note that from Theorem 3, if  $x \in \text{Core}(N, \nu)$ , then  $x_i = 0$  for any agent that does not belong to some effective coalition. Therefore, as  $E^*$  is included in all effective coalition,  $x_i = 0$ , for all  $i \in N \setminus E^*$ . □

**Remark 1** It is possible that the coalition in Eq. (1) does not contain any agent, i.e.,  $E^* = \emptyset$ . In this case,  $\nu(N) = 0$  and  $\text{Core}(N, \nu) = \{(0, 0, \dots, 0)\}$ .

Our next result shows that irreducible and elementary *mcst* games with revenues always have a non-empty  $r$ -core.

**Theorem 4** *The  $r$ -core of every irreducible and elementary mcst game with revenues is non-empty.*

**Proof** Let  $(N, \nu)$  be an irreducible and elementary *mcst* game with revenues. We show the non-emptiness by providing a particular allocation in the  $r$ -core. As the coalition provided by Eq. (1),  $E^*$ , is effective, then

$$\nu(N) = \pi(E^*) = \sum_{i \in H_*^0} (b_i - \kappa_1) + \sum_{\rho \neq 0, B_\rho > \kappa_2} (B_\rho - \kappa_2),$$

because  $B_\rho - \kappa_2$  represents the net revenue obtained by connecting only the agents in  $H^\rho$  with strictly positive contributions (i.e.,  $H_*^\rho$ ) to the source.

Then, we consider

$$x_i = \begin{cases} b_i - \kappa_1 & \text{if } i \in H_*^0 \\ \text{CEL}_i(E_\rho, D^\rho) & \text{if } \rho \neq 0, B_\rho > \kappa_2, i \in H_*^\rho \\ 0 & \text{in other case} \end{cases}, \tag{2}$$

where

$$E_\rho = (\kappa_1 - \kappa_2) + \sum_{i \in H_*^\rho} (b_i - \kappa_1)$$

and  $D^\rho$  is the vector  $D^\rho = (b_i - \kappa_1)_{i \in H_*^\rho}$ . Finally, CEL denotes the Constrained Equal Losses rule:<sup>3</sup>

$$CEL_i(E_\rho, D^\rho) = \max \{D_i^\rho - \mu, 0\} \quad \mu \text{ such that } \sum_{i \in H_*^\rho} CEL_i(E_\rho, D^\rho) = E_\rho.$$

Let us prove that  $x$  belongs to the  $r$ -core. Suppose, by contradiction, that  $x \notin \text{Core}(N, v)$ . Then, there exists a coalition  $S \subseteq N$ , such that

$$v(S) = \pi(S') > \sum_{i \in S} x_i$$

for some  $S' \subseteq S$ , that implies

$$\pi(S') > \sum_{i \in S'} x_i.$$

Note that

$$\pi(S') = \pi(S' \cap H^0) + \sum_{\rho \neq 0} \pi(S' \cap H^\rho).$$

Since  $\pi(S' \cap H^0) = \sum_{i \in S' \cap H^0} (b_i - \kappa_1) \leq \sum_{i \in S' \cap H^0} x_i$ , there must exist some  $\rho \neq 0$  for which:

$$\begin{aligned} \pi(S' \cap H^\rho) &= (\kappa_1 - \kappa_2) + \sum_{i \in S' \cap H^\rho} (b_i - \kappa_1) > \sum_{i \in S' \cap H^\rho} x_i \\ &= \alpha (\kappa_1 - \kappa_2) + \sum_{i \in S' \cap H^\rho} (b_i - \kappa_1) \end{aligned}$$

with  $\alpha \leq 1$ , as  $S' \cap H^\rho \subseteq H_*^\rho$  and the extra cost  $(\kappa_1 - \kappa_2)$  should be shared by fewer individuals. Since  $\kappa_2 > \kappa_1$ , this leads to a contradiction. Thus,  $x$  belongs to the  $r$ -core, proving the non-emptiness of this set.  $\square$

The result in Theorem 4 is significant, because it guarantees that no coalition has an incentive to deviate.

### 4 Non-elementary *mcst* problems

Working with elementary *mcst* problems may appear as a strict restriction, but as proved in Bogomolnaia and Moulin (2010), “core stability, cost monotonicity, and several other properties extend automatically from the set of elementary cost matrices

<sup>3</sup> The Constrained Equal Losses rule is a claims rule suggested by Maimonides in the twelfth century (see Aumann and Maschler 1985).

to all matrices"; that is, to all *mcst* problems. It is not immediate, however, that in *mcst* games with revenues the non-emptiness of the *r*-core extends from elementary to general problems. Our next result proves the non-emptiness in general irreducible *mcst* games with revenues, by reducing the number of different connection costs.

**Theorem 5** *The r-core of every irreducible mcst game with revenues is non-empty.*

**Proof** We first proceed with *mcst* problems having three different connection costs.

**1. Reducing the number of different connection costs**

Suppose that the *mcst* problem  $(N_\omega, \mathbf{C})$  is irreducible and it has three different connection costs,  $0 \leq \kappa_1 < \kappa_2 < \kappa_3$ . As in Theorem 1, we begin by defining the equivalence relation on  $N_\omega$ :

$$i, j \in N_\omega, \quad i \sim j \iff i = j \text{ or } c_{ij} = \kappa_1.$$

Let  $\mathcal{H}$  denote the set of equivalence classes in the quotient set, i.e., the partition obtained in Theorem 1,  $H^\rho, \rho = 0, 1, 2, \dots, m$ . Given two classes  $H^\alpha, H^\beta \in \mathcal{H}$ , we translate the cost between nodes to the cost between classes:

- If  $i \in H^\alpha$  and  $j \in H^\beta$ , then for any  $i' \in H^\alpha, c_{i'j} = \max \{c_{i'j}, c_{ij}\} = \max \{\kappa_1, c_{ij}\} = c_{ij}$  due to irreducibility of the cost matrix. Analogously, for any  $j' \in H^\beta, c_{ij'} = \max \{c_{ij}, c_{jj'}\} = \max \{c_{ij}, \kappa_1\} = c_{ij}$ .
- Consequently, for any  $i, i' \in H^\alpha, j, j' \in H^\beta, c_{ij} = c_{i'j'}$ .
- Given  $H^\alpha, H^\beta \in \mathcal{H}, \alpha \neq \beta, c_{\alpha\beta} = c_{ij}$  for  $i \in H^\alpha, j \in H^\beta$ .
- $c_{\alpha\beta} \in \{\kappa_2, \kappa_3\}$ .

Moreover, we associate to every class  $H^\rho, \rho \neq 0$ , the revenue

$$B_\rho = \sum_{i \in H_*^\rho} b_i - \kappa_1 (|H_*^\rho| - 1) \quad \text{where } H_*^\rho = \{i \in H^\rho : b_i > \kappa_1\}.$$

Then, we have a new *mcst* problem with revenues defined by

- The *source*:  $H^0$ .
- The *agents*:  $H^\rho, \rho = 1, 2, \dots, m$ .
- The *connection costs*:  $c_{\alpha\beta} \in \{\kappa_2, \kappa_3\}, \alpha \neq \beta, \alpha, \beta \in \{0, 1, 2, \dots, m\}$ .
- The *revenues*:  $B_\rho, \rho = 1, 2, \dots, m$ .

Obviously, this problem is elementary, since only two possible connection costs appear between agents:  $0 < \kappa_2 < \kappa_3$ . Let us see that the problem is also irreducible. Consider  $\alpha, \beta, \gamma \in \{0, 1, 2, \dots, m\}$ . From Corollary 1, we need to prove that

$$c_{\alpha\gamma} \leq \max \{c_{\alpha\beta}, c_{\beta\gamma}\}.$$

We know that

- $c_{\alpha\gamma} = c_{ij} \quad i \in H^\alpha, j \in H^\gamma.$
- $c_{\alpha\beta} = c_{ik} \quad i \in H^\alpha, k \in H^\beta.$
- $c_{\beta\gamma} = c_{kj} \quad k \in H^\beta, j \in H^\gamma.$

As the *mcst* problem  $(N_\omega, \mathbf{C})$  is irreducible,  $c_{ij} \leq \max \{c_{ik}, c_{kj}\}$  and the ultrametric condition is fulfilled in the new problem. Moreover, we note that the new revenues game  $(\mathcal{H}, v_{\mathcal{H}})$  obviously fulfills that

$$v_{\mathcal{H}}(\mathcal{H}) + v(H_*^0) = v(N) \quad \text{where} \quad v(H_*^0) = \sum_{i \in H_*^0} (b_i - \kappa_1).$$

**2. Recursive applying of the elementary problem result**

Hence, by applying Theorem 4 to the new problem, we conclude that the *r*-core of the *mcst* game with revenues  $(\mathcal{H}, v_{\mathcal{H}})$  is non-empty; that is, we can assign to every agent (subset)  $H^\rho \in \mathcal{H}$  an amount  $x^\rho$ , such that:

$$\sum_{\rho=1}^m x^\rho = v_{\mathcal{H}}(\mathcal{H}), \quad \sum_{H^\rho \in \mathcal{S}} x^\rho \geq v_{\mathcal{H}}(\mathcal{S}) \quad \text{for all } \mathcal{S} \subseteq \mathcal{H}.$$

The next step is to decide how to distribute the allocation  $x^\rho$  among the individuals in  $H^\rho$ . We distinguish the following cases:

- (a)  $i \in H^0$  :  

$$x_i = \max\{b_i - \kappa_1, 0\}.$$
- (b)  $i \in H^\rho, \rho \neq 0$  :  

$$x_i = \max\{b_i - \kappa_1 - \varepsilon, 0\}, \varepsilon \text{ such that } \sum_{i \in H^\rho} x_i = x^\rho.$$

Now, given  $S \subseteq N$ , by observing that

$$v(S) = \sum_{i=0}^m v(S \cap H^\rho),$$

it is clear (by reasoning as in Theorem 4) that  $v(S) \leq \sum_{i \in S} x_i$  and the provided allocation belongs to the *r*-core of the initial *mcst* game with revenues, proving its non-emptiness.

Finally, it is possible to replicate the previous argument if we have four (or more) different connection costs, in each step reducing the number of different connection costs. □

**5 Final comments**

While it can be argued that cost matrices in *mcst* problems are rarely irreducible, it is possible to derive the irreducible cost matrix  $\mathbf{C}^*$  from the original cost matrix  $\mathbf{C}$ . Then,  $\mathbf{C}^*$  can be used to define the *mcst* game with revenues.

However, the use of irreducible costs has faced criticism. This approach is often labeled as reductionist, as it disregards interpersonal differences in relevant connecting costs (Bogomolnaia and Moulin 2010). By reducing the problem to irreducible costs,

valuable information about alternative paths or edges—which could be crucial in dynamic or uncertain environments—may be lost. As highlighted in Bogomolnaia and Moulin (2010), the process of deriving  $\mathbf{C}^*$  from  $\mathbf{C}$  involves a significant reduction in data: while the original matrix  $\mathbf{C}$  contains  $\frac{n(n+1)}{2}$  independent entries, the irreducible matrix  $\mathbf{C}^*$  retains only  $n$  of these, corresponding to the costs of edges present in some optimal spanning tree. This drastic pruning discards much of the information that could be relevant for ensuring fairness in cost allocation.

Despite these criticisms, irreducible costs offer notable advantages in certain scenarios. For instance, when analyzing *mcst* problems with revenues involving more than two distinct connection costs, the core of the associated cooperative game (the *r*-core) is typically empty. Using the irreducible costs instead of the original connection costs, it becomes possible to identify a feasible allocation of the net revenue that discourages any coalition from leaving the group. Moreover, the use of irreducible costs implies that agents agree to bear the highest connection cost along their optimal path to the source, fostering cooperation among them. Finally, irreducible costs simplify the *mcst* problem by eliminating redundant or dominated edges, making it more computationally tractable. Alternative reductions, such as cycle-complete cost matrices (Trudeau 2012) or quasi-ultrametric cost structures, could also be explored to test the robustness of *r*-core non-emptiness beyond strict irreducibility.

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**Data availability** No new data were generated or analyzed in this study.

## Declarations

**Conflict of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. This article does not contain any studies with human participants or animals performed by any of the authors.

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