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Non-emptiness of the core of *mcst* games with revenues

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ABSTRACT

The minimum cost spanning tree (*mcst*) problem analyzes how to efficiently connect geographically dispersed agents to a common source. Estévez-Fernández and Reijnierse (2014) study *mcst* problems involving revenues, in which the agents benefit from being connected to the source. A key feature of their framework is that only certain coalitions, called *effective coalitions*, make sense to form. Estévez-Fernández and Reijnierse (2014) show that the core in cost–revenue *mcst* games is typically empty. In this paper, we present a necessary condition that *mcst* games with revenues need to fulfill to ensure the non-emptiness of the core: the existence of at least one agent with *veto power*, i.e., an agent whose participation is required in every effective coalition. Additionally, we extend the sufficient condition in Estévez-Fernández and Reijnierse (2014) for elementary cost *mcst* problems.

1. Introduction

Consider a scenario in which individuals situated in different locations seek to connect to a source to access a good or service. Each connection between individuals or to the source entails a fixed cost. Furthermore, it is assumed that individuals are indifferent to whether the connection is direct to the source or indirect through other individuals. This situation, known as the minimum cost spanning tree (*mcst*) problem, is widely used to analyze real-world scenarios, including the design of telephone and cable TV networks or water supply systems.

Several methods exist to determine a network that connects all agents to the source while minimizing total connection costs, resulting in an optimal tree. A key question that arises is how to allocate this minimum cost among the individuals, with numerous proposals explored in the literature (see, for instance, Bogomolnaia and Moulin (2010) and Bergantiños and Vidal-Puga (2021) for definitions and comparative analyses).

Estévez-Fernández and Reijnierse (2014) study the *mcst* problem under the consideration of revenues generated by the service provided. They analyze a scenario where a group of agents cooperates to reduce the (potentially nonlinear) costs associated with service provision. This raises the question of how the net revenues generated through cooperation should be distributed among the agents. Estévez-Fernández and Reijnierse (2014) show that the core of the associated cooperative game may be empty, even when connection costs take only two possible values: low and high. Such scenarios are known as elementary cost problems.

A distinctive feature of *mcst* problems with revenues is that a coalition may benefit from excluding some of its members from accessing the service that generates revenue. Consequently, certain individuals may remain disconnected in the optimal cost–revenue spanning tree. Estévez-Fernández and Reijnierse (2014) derive a sufficient condition to ensure the non-emptiness of the core in elementary cost *mcst* revenue games. Their condition stipulates that all individuals are connected within the optimal cost–revenue spanning tree. However, this condition is not necessary, and it overlooks a key characteristic of *mcst* problems with revenues: the possibility that some agents may remain unconnected.

To identify other classes of *mcst* problems with revenues that have a non-empty core, this paper introduces the concept of maximal effective coalition. Using this concept, we obtain a necessary condition for the core to be non-empty: at least one agent must belong to every maximal effective coalition. This result applies to general *mcst* problems with revenues. Next, following the framework established by Estévez-Fernández and Reijnierse (2014), we focus on elementary cost problems and derive an alternative (more general) sufficient condition.

The remainder of the paper is organized as follows: Section 2 presents the main definitions and it introduces *mcst* problems with revenues. Section 3 contains our main results, including necessary, and sufficient conditions that ensure the non-emptiness of the core. The paper concludes with some final remarks.

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2. Preliminaries

An *mcst* problem involves a finite set of agents, $N = \{1, 2, \dots, n\}$, $n \geq 2$, who want to connect to a source ω. The set of agents and the source is denoted by $N_\omega = N \cup \{\omega\}$. Agents and the source are potentially connected by links (or edges), where for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the link $e_{ij} = (i, j)$ connecting $i, j \in N_\omega$. As usually, we assume $c_{ii} = 0$ and $c_{ij} = c_{ji}$, for all $i, j \in N_\omega$. The *mcst* problem is formally represented by the pair $(N_\omega, \mathbf{C})$, where $\mathbf{C} = [c_{ij}]_{(n+1)\times(n+1)}$ is the non-negative cost matrix. An efficient method to identify an spanning tree with minimum cost is provided by Prim's algorithm (Prim, 1957).

An extensively studied family of *mcst* problems is the class of elementary cost problems, where the cost of any edge can only take two values, low cost or high cost; that is, for all $i, j \in N_\omega$, $i \neq j$, $c_{ij} \in \{\kappa_1, \kappa_2\}$, with $0 \leq \kappa_1 < \kappa_2$. These problems are fundamental for analyzing more complex scenarios, since they form a basis for all *mcst* problems (see Bogomolnaia and Moulin, 2010).

Estévez-Fernández and Reijnierse (2014) extend the *mcst* problem to incorporate revenues, introducing what is known as the *mcst* problem with revenues. In this framework, consider an *mcst* problem $(N_\omega, \mathbf{C})$, where a subset of agents $S \subseteq N$ connected to the source ω can cooperate to generate higher joint revenue. When a coalition $S \subseteq N$ chooses to cooperate, this collaboration results in additive revenue while incurring a non-linear cost associated with connecting individuals in S to the source ω.

Formally, let $b_i \in \mathbb{R}_+$ denote the revenue generated by agent $i \in N$ if i gets the service; that is, if agent i is connected to the source ω, either directly or indirectly (through other agents). The vector of revenues is represented as $b = (b_1, b_2, \dots, b_n)$. The total revenue that a coalition $S \subseteq N$ can achieve through cooperation is $\sum_{i\in S} b_i$, while the net revenue of coalition S is given by $\pi(S) = \sum_{i\in S} b_i - c(S)$. Here, $c(S)$ represents the (minimum) cost of connecting the members of coalition S to the source. Given the revenue structure of the game, a coalition $S \subseteq N$ may achieve higher profits by excluding certain members from connecting to the source ω. Under this assumption, the worth of a coalition $S \subseteq N$ in the *mcst* problem $\left(N_\omega, \mathbf{C}\right)$ with revenues $b = (b_1, b_2, \dots, b_n)$ is defined as:

$$\upsilon(S) = \max\{\pi(R), \quad R \subseteq S\}.$$

By using this function the TU-game associated to an *mcst* problem with revenues is defined.

Definition 1. Let $\left(N_\omega, \mathbf{C}\right)$ be an *mcst* problem with revenues b. The associated TU-game is the pair (N, υ).

The concept of *effective coalition* identifies subsets of agents that maximize net revenues.

Definition 2. Let $\left(N_\omega, \mathbf{C}\right)$ be an *mcst* problem with revenues $b = (b_1, b_2, \dots, b_n)$. An **effective coalition** is defined as a subset of agents $E \subseteq N$, such that $\upsilon(N) = \pi(E)$. The set of all effective coalitions will be denoted by $\mathcal{E}_{ff}(\upsilon)$.

As illustrated in Estévez-Fernández and Reijnierse (2014), multiple effective coalitions may exist. Let us denote by $N^\cup$ and $N^\cap$, respectively, the union and intersection of all effective coalitions:

$$N^\cup = \bigcup_{E\in\mathcal{E}_{ff}(\upsilon)} E \qquad N^\cap = \bigcap_{E\in\mathcal{E}_{ff}(\upsilon)} E.$$

When agents agree to cooperate, they must determine how to share the net revenue they generate. Ensuring that a sharing $x = \left(x_1, x_2, \dots, x_n\right)$ of the total worth $\upsilon(N)$ belongs to the core of the game, denoted as $Core(\upsilon)$ (which we refer to as the *r*-core, where the letter *r* stands for revenues), guarantees stability and prevents any agent or coalition from deviating from the cooperation:

$$x(N) \equiv \sum_{i\in N} x_i = \upsilon(N) \quad \text{and for any subset } S \subseteq N, \quad x(S) \equiv \sum_{i\in S} x_i \geq \upsilon(S).$$

Remark 1. As $\pi(\emptyset) = 0$, then $\upsilon(S) \geq 0$ for any coalition $S \subseteq N$. This implies that any allocation $x = \left(x_1, x_2, \dots, x_n\right)$ in the *r*-core fulfills $x_i \geq 0$, for all $i \in N$.

Before presenting our results, we define non-trivial problems as those with strictly positive net revenue.

Definition 3. An *mcst* game with revenues (N, υ) is said to be **non-trivial** if $\upsilon(N) > 0$.

Remark 2. For *trivial mcst* games with revenues, it is always true that $\upsilon(S) = 0$ for any coalition $S \subseteq N$. Consequently, the *r-core* of a trivial problem is non-empty:

$$Core(\upsilon) = \{(0, 0, \dots, 0)\}.$$

Thus, we focus exclusively on non-trivial problems.

3. Non-empty *r -core*

In general, the *r-core* may be empty even for the particular class of elementary cost *mcst* problems with revenues (Estévez-Fernández and Reijnierse, 2014). In this section we analyze necessary, and sufficient conditions regarding the non-emptiness of the *r-core*. From now on, we will consider only *non-trivial* problems, i.e., problems such that $\upsilon(N) > 0$.

3.1. A general necessary condition

Theorem 1. *Every non-trivial mcst game with revenues (N, υ) having a non-empty r-core satisfies $N^\cap \neq \emptyset$.*

Proof. Let $x \in Core(\upsilon)$. Assume, for the sake of contradiction, that $N^\cap = \emptyset$. Let E denote one of the effective coalitions. As the intersection of all effective coalitions is empty, for every agent $i \in E$ there is an effective coalition $E' \neq E$ such that $i \notin E'$. Suppose $x_i > 0$. Then,

$$\upsilon(N) = x(N) \quad \Rightarrow \quad x(N \setminus \{i\}) = \upsilon(N) - x_i < \upsilon(N)$$

On the other hand, as $i \notin E'$, we have:

$$\upsilon(N) = \upsilon\left(E'\right) \leq x\left(E'\right) \leq x(N \setminus \{i\}) = \upsilon(N) - x_i < \upsilon(N)$$

which is a contradiction. Therefore, $x_i = 0$. Repeating this process for every $i \in E$, we conclude that $x(E) = 0$. This contradicts the assumption that $x \in Core(\upsilon)$, since E is an effective coalition and the problem is non-trivial, so $x(E) \geq \upsilon(E) = \upsilon(N) > 0$. Thus, $N^\cap \neq \emptyset$. ■

The above result resembles a well-known property of simple games that ensures the core is nonempty if and only if there is at least one *veto player* (see Owen, 1982). Interestingly, most *mcst* games with revenues (N, υ) are not simple games. Therefore, it is only necessary that there exists at least one agent without whom the maximum revenue cannot be obtained, and it is not difficult to find an example proving that this condition is not sufficient to guarantee the non-emptiness of the *r-core*. On the other hand, the result in Theorem 1 provides a way to prove that the *r-core* is empty. The following proposition establishes additional properties.

Proposition 1. *Let (N, υ) be a non-trivial mcst game with revenues. If $S \cap E = \emptyset$ for some effective coalition E, then $\upsilon(S) = 0$. Moreover, if $x \in Core(\upsilon)$, then $x_i = 0$ for all $i \notin N^\cap$.*

Proof. Suppose $S \cap E = \emptyset$ for some effective coalition E, and $\upsilon(S) > 0$. From its definition, $\upsilon(S) = \pi(T) > 0$, for some $T \subseteq S$. Therefore, $T \cap E = \emptyset$. Now consider the coalition $E \cup T$:

$$\pi(E \cup T) = \sum_{i\in E\cup T} b_i - c(E \cup T) \geq \sum_{i\in E\cup T} b_i - c(E) - c(T) =$$

$= \pi(E)+\pi(T) > \pi(E) = v(N)$

which contradicts the definition of $v(N)$. Hence, $v(S) = 0$.

Consider now $x \in Core(v)$. For any $i \notin N^{\cap}$, there exists some effective coalition E such that $i \notin E$. Therefore,

$$x(E) \geq v(E) = v(N) \quad \Rightarrow \quad x(N \setminus E) = 0 \quad \Rightarrow \quad x_j = 0 \text{ for all } j \notin E$$

since $x_j \geq 0$ for all $j \in N$. ■

3.2. Sufficient conditions for elementary cost–revenue games

Estévez-Fernández and Reijnierse (2014) proved that in elementary cost *mcst* problems with revenues, whenever N is an effective coalition the *r-core* is non-empty. In this section we present a more general sufficient condition.

Theorem 2. *Every non-trivial elementary cost mcst game with revenues (N, v) such that $N^{\cup}$ is effective has a non-empty r-core.*

Obviously, if N is effective, then $N^{\cup} = N$, so Theorem 2 generalizes the result in Estévez-Fernández and Reijnierse (2014). To prove this result, some prior notation and results are required. First, we introduce an auxiliary function. If $S \subseteq N$, let:

$$\mu(S) = \begin{cases} \max\left\{v(S \cup H) : H \cap N^{\cup} = \emptyset\right\} & \text{if } S \cap E \neq \emptyset, \text{for all} E \in \mathcal{E}_{ff}(v) \\ 0 & \text{otherwise} \end{cases}$$

This function serves as an upper bound for $v(S)$ and a lower bound for $x(S)$ for any $x \in Core(v)$.

Proposition 2. *Given a non-trivial mcst game with revenues (N, v), for every $S \subseteq N$:*

(a) $\mu(S) \geq v(S)$
(b) $x \in Core(v) \quad \Rightarrow \quad x(S) \geq \mu(S)$

Proof (*a*). If $S \cap E = \emptyset$, for some $E \in \mathcal{E}_{ff}(v)$, from Proposition 1, $v(S) = \mu(S) = 0$. If $S \cap E \neq \emptyset$ for all $E \in \mathcal{E}_{ff}(v)$, then

$$\mu(S) = \max\left\{v(S \cup T) : T \cap N^{\cup} = \emptyset\right\} \geq v\left(S \cup \emptyset\right) = v(S).$$

(*b*) If $S \cap E = \emptyset$, for some $E \in \mathcal{E}_{ff}(v)$, then $\mu(S) = 0 \leq x(S)$. On the other hand, if $S \cap E \neq \emptyset$ for all $E \in \mathcal{E}_{ff}(v)$, then $\mu(S) = v(S \cup H)$ for some coalition H such that $H \cap N^{\cup} = \emptyset$. Then, from Proposition 1, $x_j = 0$ for all $j \in H$, and we get

$$x(S) = x(S \cup H) \geq v(S \cup H) = \mu(S).$$

For elementary cost *mcst* problems additional properties are obtained. Subiza et al. (2016) show that every elementary cost *mcst* problem $(N_{\omega}, \mathbf{C})$ the set of agents can be split as the union of disjoint sub-problems called *simple components*; i.e., $N = \bigcup_{M \in \mathcal{M}} M$, such that $c(N) = \sum_{M \in \mathcal{M}} c(M)$, and for all $M \in \mathcal{M}$:

(1) There exists at most one individual $i \in M$ such that $c_{i\omega} = \kappa_1$.
(2) For all $i \in M$, $\min\left\{c_{ij} : j \in M, j \neq i\right\} = \kappa_1$.
(3) $c(M) = \begin{cases} m\kappa_1 & \text{if } c_{i\omega} = \kappa_1 \text{ for some } i \in M \\ m\kappa_1 + (\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$
where $m = |M|$.

It is usual, especially in elementary cost problems, that several trees solve the minimization cost problem. From now on, for elementary cost *mcst* problems, we use a minimum cost spanning tree, which we denote by m, with the following characteristics[1]:

(M1) $c_{im(i)} = \kappa_2 \Rightarrow m(i) = \omega$
that is, agents only use the high-cost connection to join the source; and
(M2) $c_{i\omega} = \kappa_1 \Rightarrow m(i) = \omega$
i.e., agents with a low cost to the source connect the source directly.

For any $i \in N$, the agent to which i connects in spanning tree m is denoted by $m(i)$.

Proposition 3. *Let (N, v) be a non-trivial elementary cost mcst game with revenues such that $N^{\cup}$ is effective. Then, for all $i \in N$:*

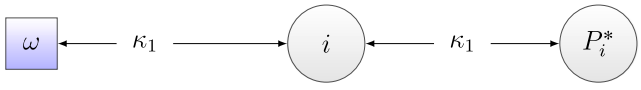
(a) $b_i - \kappa_2 \leq \mu(\{i\})$.
(b) $\mu(\{i\}) \leq \max\left\{b_i - \kappa_1, 0\right\}$.

Proof. Part *(a)* comes directly from $c_{i\omega} \leq \kappa_2$, and $v(\{i\}) = \max\left\{b_i - c_{i\omega}, 0\right\}$, therefore $\mu(\{i\}) \geq v(\{i\}) \geq b_i - c_{i\omega} \geq b_i - \kappa_2$.

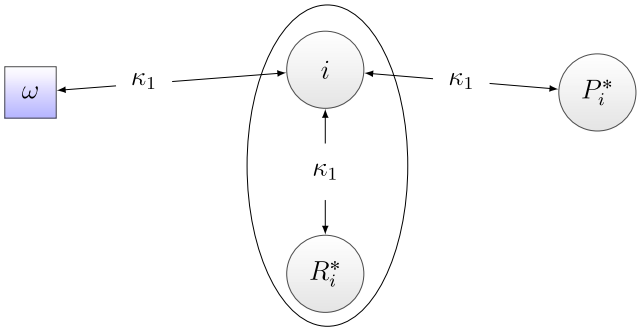
To prove part *(b)*, if $\mu(\{i\}) = 0$, it follows directly. If $\mu(\{i\}) > 0$ for some $i \in N$, there is $T_i \subseteq N$, $T_i \cap N^{\cup} = \emptyset$, such that $\mu(\{i\}) = v\left(T_i \cup \{i\}\right) = \pi\left(R_i\right)$, $R_i \subseteq T_i \cup \{i\}$. If $i \notin R_i$, then $v\left(R_i\right) = 0$ (since $R_i \cap N^{\cup} = \emptyset$), so $\pi\left(R_i\right) \leq 0$ and $\mu(\{i\}) > 0$ is not possible. Therefore, $i \in R_i$.

Let m be a minimum cost tree satisfying characteristics (M1) and (M2) that connects agents in $N^{\cup}$ to the source and let P_i be the simple component that contains agent i. If we suppose $\mu(\{i\}) = \pi\left(R_i\right) > \max\left\{b_i - \kappa_1, 0\right\}$, in P_i, we show that replacing this agent by coalition R_i we can obtain a greater joint revenue. We distinguish the following cases:

(1) if $P_i = \{i\}$, we can directly replace the agent i with the coalition R_i to get a higher revenue.
(2) If i joins directly the source at low cost, the configuration in P_i is as follows[2]:



where $P_i^* = P_i \setminus \{i\}$ and the net revenue is $\pi\left(P_i\right) = b_i - \kappa_1 + \sum_{j \in P_i,\, j \neq i} \left(b_j - \kappa_1\right)$. The configuration in $P_i \cup R_i$ is:



where $R_i^* = R_i \setminus \{i\}$. Now, the net revenue is $\pi\left(P_i \cup R_i\right) = \pi\left(R_i\right) + \sum_{j \in P_i,\, j \neq i} \left(b_j - \kappa_1\right)$ so, $\pi\left(P_i\right) < \pi\left(P_i \cup R_i\right)$.

[1] In the elementary cost *mcst* context, it is obvious that we can always select a cost-optimal spanning tree featuring these properties.

[2] Note that the nodes in this configuration do not necessarily are in a row, since P_{i1}^* can be an arbitrary tress in which all connections have the minimal cost κ_1.

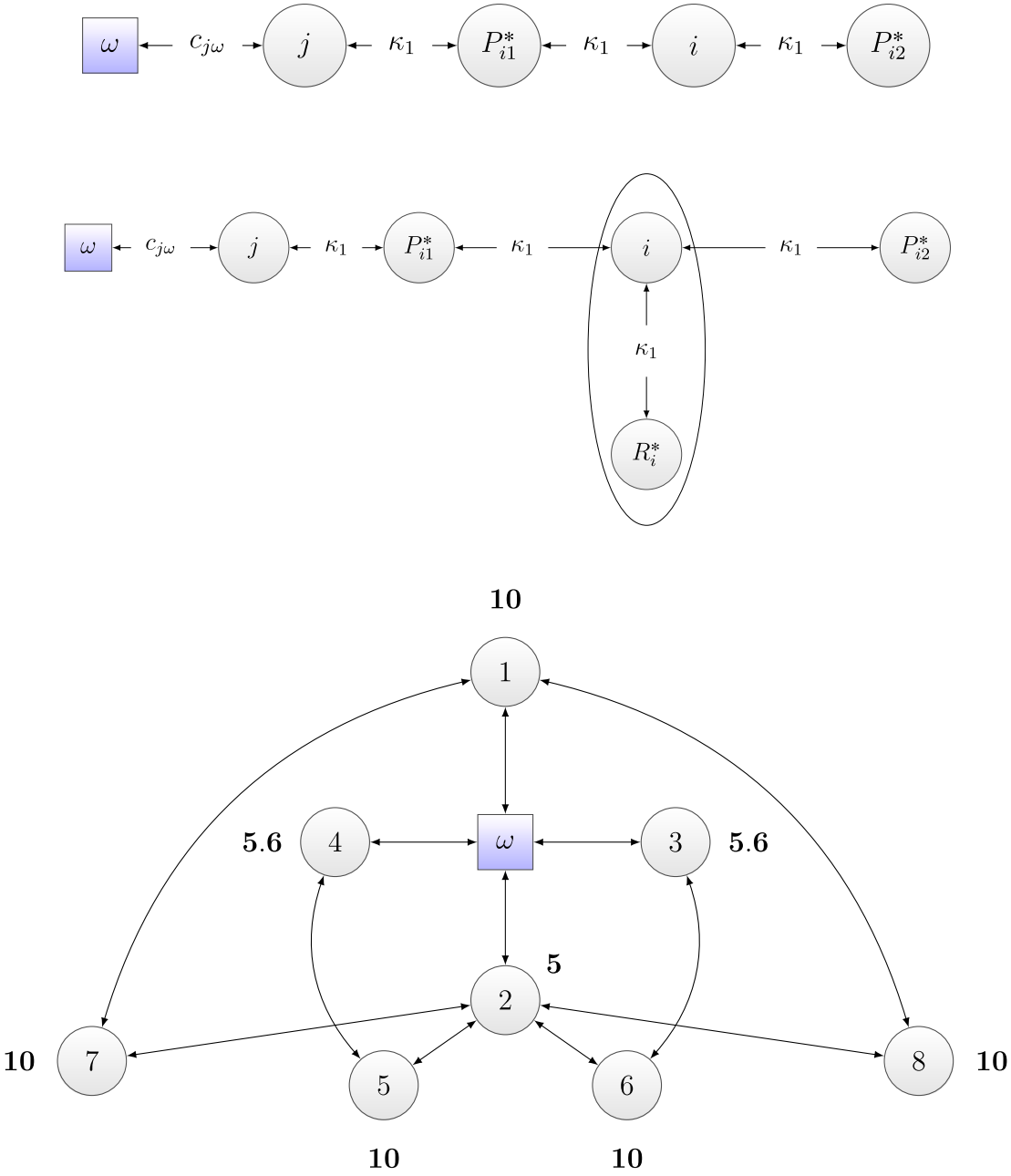


Fig. 1. Elementary cost *mcst* problem with revenues. The cost of each link appearing in the picture is $c_{ij} = 6$, whereas the cost of the undrawn links is $c_{ij} = 1000$. The boldfaced numbers at the nodes represent the revenues.

(3) If i does not connect directly with the source the new configurations are as follows and we can reason as in the previous case.

where P^*_{i1} denotes the agents in this simple component that "precede" agent i in the path to the source, and P^*_{i2} denotes those agents that connect to the source through agent i. Again we obtain, $\pi\left(P_i\right) < \pi\left(P_i \cup R_i\right)$.

Since any other simple component in $N^{\cup}$ remains unchanged, $\pi\left(N^{\cup} \cup R_i\right) > \pi\left(N^{\cup}\right)$, a contradiction. ■

Before stating our sufficiency result, we introduce the idea of the proof through the following example.

Example 1. Consider the *mcst* problem with revenues represented in Fig. 1.

In this case there is only one effective coalition, $E = \{1, 3, 4, 5, 6, 7, 8\}$, and $v(N) = \pi(E) = 19.2$. The characteristic function μ is defined by $\mu(S) = v(S \cup \{2\})$, for all $S \subseteq E$. In particular,

$$\mu(1) = 4 \qquad \mu(2) = \mu(3) = \mu(4) = 0 \qquad \mu(5) = \mu(6) = \mu(7) = \mu(8) = 3.$$

Now we build a modified problem $\left(N', v'\right)$ in the following way (see Fig. 2):

(a) We eliminate agents that are not in the effective coalition.
(b) If $i \in E$, but $\mu(i) = 0$, we eliminate agent i (assigning a payoff $x_i = 0$ to this agent) and adjust the revenues of agents that connect to i so that the "joint net revenue" does not change.

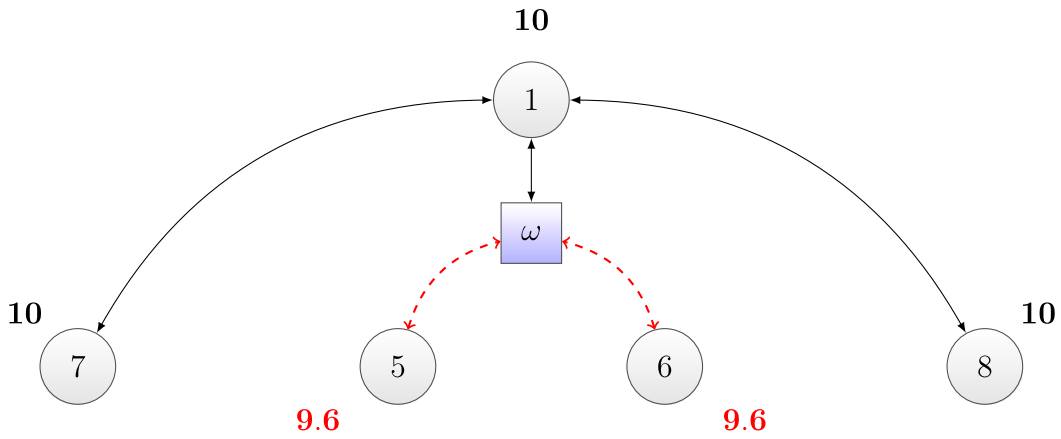


Fig. 2. Modified elementary cost *mcst* problem with revenues. The cost of each link (solid and dashed lines) in the picture is $c_{ij} = 6$, whereas the cost of the undrawn links is $c_{ij} = 1000$.

Then, $N' = \{1, 5, 6, 7, 8\}$. Agent 3 has been removed from the network, and the revenue of agent 6 (the one that connects to agent 3) has been modified so that the joint net revenue of these agents remains unchanged: $b'_6 = 10 + 5.6 - 6 = 9.6$. An analogous change is made involving agents 4 and 5. The new problem (N', v') satisfies the sufficient condition in Estévez-Fernández and Reijnierse (2014), so the non-emptiness of the *r*-core can be ensured, $Core\,(v') \neq \emptyset$. By assigning a payoff $x_i = 0$ to any agent outside N' we obtain an allocation in the *r*-core of the original problem.

We are now ready to prove that, for the elementary cost *mcst* games with revenues, if $N^{\cup}$ is effective the *r*-core is non-empty.

Proof of Theorem 2. As $N^{\cup}$ is effective

$$v(N) = v(N^{\cup}) = \pi(N^{\cup}) = \sum_{i \in N^{\cup}} b_i - c(N^{\cup}) = \sum_{t=1}^{q} \left(\sum_{i \in N_t} b_i - c\left(N_t\right) \right).$$

where q is the number of simple components in the *mcst* problem $\left(N^{\cup}_{\omega}, \mathbf{C}\,|_{N^{\cup}}\right)$ and N_t, $t = 1, 2, \ldots, q$ denote these simple components. We prove several claims.

Claim 1. *If $S \subseteq N \setminus N^{\cup}$ and $T \subseteq N^{\cup}$, $\pi(S \cup T) \leq \pi(T)$.*

We first define an equivalence binary relation among the agents in S. Given $i, j \in S$

$$i \sim_S j \Leftrightarrow \exists\, i_1, i_2, \ldots, i_t \in S, \text{ such that } i_1 = i,\ i_t = j \text{ and } c_{i_k i_{k+1}} = \kappa_1, \quad \forall\, k = 1, 2, \ldots, t-1$$

that is, there is a chain in S joining i and j with all connections having low cost. This equivalence relation divides coalition S in several equivalence classes such that:

(a) elements within the same class can be connected (directly or indirectly) with low-cost connections;
(b) elements within different classes only can be connected at high-cost.

On the other hand, the classes in S can be differentiated in three types:

TYPE S^1 : $\forall i \in S^1$, $c_{i\omega} = \kappa_2$; and there are $i \in S^1$, $j \in N^{\cup}$ such that $c_{ij} = \kappa_1$;
TYPE S^2 : $\forall i \in S^2$, $c_{ij} = \kappa_2$, for all $j \in N^{\cup} \cup \{\omega\}$;
TYPE S^3 : there is $i \in S^3$ such that $c_{i\omega} = \kappa_1$.

For simplicity, we will suppose that there is only one subcoalition of each type, so

$$S = S^1 \cup S^2 \cup S^3.$$

Otherwise, we can replicate the argument as many times as necessary for each type (see Fig. 3).

The cost of connecting coalition S to the source is:

$$c(S) = \left|S^1\right| \kappa_1 + \left(\kappa_2 - \kappa_1\right) + \left|S^2\right| \kappa_1 + \left(\kappa_2 - \kappa_1\right) + \left|S^3\right| \kappa_1$$

If S joins $N^{\cup}$, the cost of the new coalition is

$$c\left(S \cup N^{\cup}\right) = c\left(N^{\cup}\right) + \left|S^1\right| \kappa_1 + \left|S^2\right| \kappa_1 + \left(\kappa_2 - \kappa_1\right) + \left|S^3\right| \kappa_1 - \alpha\left(\kappa_2 - \kappa_1\right)$$

where α is the number of agents in $N^{\cup}$ that reduce the cost by connecting to the source *via* S; that is, when only considering $N^{\cup}$ these agents directly connect to the source at a high-cost, and now they could connect with some agent in S at a low-cost. Then,

$$c\left(S \cup N^{\cup}\right) - c\left(N^{\cup}\right) = \left|S^1\right| \kappa_1 + \left|S^2\right| \kappa_1 + \left(\kappa_2 - \kappa_1\right) + \left|S^3\right| \kappa_1 - \alpha\left(\kappa_2 - \kappa_1\right)$$

Now we repeat the argument by replacing coalition $N^{\cup}$ by coalition $T \subseteq N^{\cup}$. Then,

$$c(S \cup T) - c(T) = \left|S^1\right| \kappa_1 + \sigma\left(\kappa_2 - \kappa_1\right) + \left|S^2\right| \kappa_1 + \left(\kappa_2 - \kappa_1\right) + \left|S^3\right| \kappa_1 - \beta\left(\kappa_2 - \kappa_1\right)$$

where β is the analogous of α when considering coalition T instead of $N^{\cup}$, and $\sigma \in \{0, 1\}$ indicates if there is or not some $j \in T$ such that $c_{ij} = \kappa_1$ for some $i \in S^1$. By construction, $\beta \leq \alpha$, so

$$c(S \cup T) - c(T) \geq c\left(S \cup N^{\cup}\right) - c\left(N^{\cup}\right) \tag{1}$$

Now we are ready to prove Claim 1. If we suppose the contrary,

$$\pi(S \cup T) > \pi(T) \Rightarrow \sum_{i \in S \cup T} b_i - c(S \cup T) > \sum_{i \in T} b_i - c(T) \Rightarrow \sum_{i \in S} b_i > c(S \cup T) - c(T)$$

From Eq. (1)

$$\sum_{i \in S} b_i > c(S \cup T) - c(T) \geq c\left(S \cup N^{\cup}\right) - c\left(N^{\cup}\right)$$

Then,

$$\pi\left(S \cup N^{\cup}\right) = \sum_{i \in S \cup N^{\cup}} b_i - c\left(S \cup N^{\cup}\right) = \sum_{i \in S \cup N^{\cup}} b_i - c\left(N^{\cup}\right) - \left(c\left(S \cup N^{\cup}\right) - c\left(N^{\cup}\right)\right) =$$

$$= \sum_{i \in N^{\cup}} b_i - c\left(N^{\cup}\right) + \sum_{i \in S} b_i - \left(c\left(S \cup N^{\cup}\right) - c\left(N^{\cup}\right)\right) > \sum_{i \in N^{\cup}} b_i - c\left(N^{\cup}\right) = \pi\left(N^{\cup}\right)$$

contradicting that $N^{\cup}$ is an effective coalition.

Claim 2. *For any $T \subseteq N^{\cup}$, $\mu(T) = v(T)$.*

If $T \cap E = \emptyset$ for some effective coalition, then $\mu(T) = v(T) = 0$. Otherwise, if $T \cap E \neq \emptyset$ for all effective coalitions, let $S \subseteq N \setminus N^{\cup}$ such that $\mu(T) = v(T \cup S)$. From the definition of the characteristic function v, there are sub-coalitions $T' \subseteq T$ and $S' \subseteq S$ such that

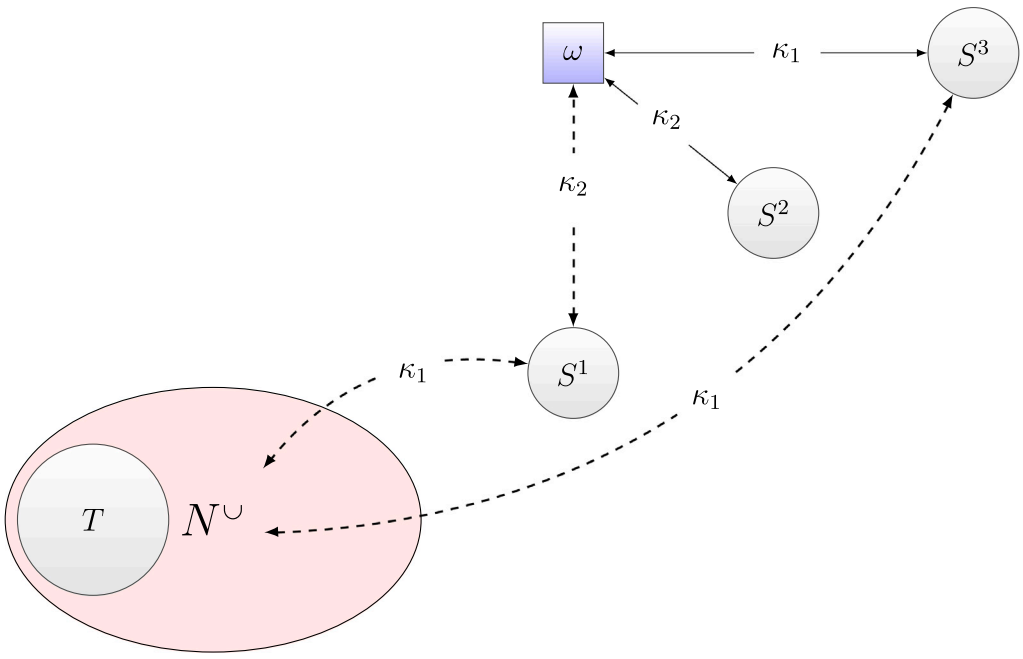


Fig. 3. Connection diagram: Solid lines indicate costs that cannot be changed. The dashed lines represent costs that could be reduced by merging S and $N^{\cup}$.

$\mu(T) = v(T \cup S) = \pi(T' \cup S')$. By applying Proposition 2 and Claim 1,

$$v(T) \leq \mu(T) = \pi(T' \cup S') \leq \pi(T') \leq v(T)$$

then all the inequalities are equal and we have $\mu(T) = v(T)$.

Given $T \subseteq N^{\cup}$, let δ_T denote the number of simple components in the *mcst* problem $\left(T_{\omega}, \mathbf{C}\big|_T\right)$ that connect to the source at high cost[3] and let θ_T denote the number of agents in T that connect at high cost (so they are directly connected to the source) in the *mcst* problem $\left(N^{\cup}_{\omega}, \mathbf{C}\big|_{N^{\cup}}\right)$.

Claim 3. *For any $T \subseteq N^{\cup}$, $\delta_T \geq \theta_T$.*

Let us suppose the contrary, $\delta_T < \theta_T$. Then there is at least one agent $i^0 \in T$ such that i^0 belongs to an expensive component in $N^{\cup}$, but belongs to a cheap component in T. This simple component may be used in $N^{\cup}$ to reduce the cost of the spanning tree, but this contradicts that $N^{\cup}$ is an effective coalition.

Now, we will define an allocation vector x that belongs to the r-core. We distinguish three types of coalitions:

(1) $T \cap N^{\cup} = \emptyset$.
In this case $v(T) = 0$ from Proposition 1 and $x(T) \geq v(T)$, since $x_i \geq 0$ for all $i \in N$.

(2) $T \subseteq N^{\cup}$.

(3) $T \cap N^{\cup} \neq \emptyset$ and $T \nsubseteq N^{\cup}$.
In this case $v(T) = \pi(T')$, for some $T' \subseteq T$. If $T' \cap N^{\cup} = \emptyset$, then $v(T) = 0$ and $x(T) \geq v(T)$. In other case, $T' = T'' \cup S$ where $T'' \subseteq N^{\cup}$ and $S \cap N^{\cup} = \emptyset$. Then, from Claim 1

$$v(T) = \pi(T') = \pi(T'' \cup S) \leq \pi(T'') \leq v(T'')$$

Then this situation becomes case (2).

Therefore, and applying Claim 2, to prove the non-emptiness of the r-core it is sufficient to obtain a vector $x \in \mathbb{R}^n_+$ such that $x(N) = v(N)$ and $\mu(T) \leq x(T)$, for all $T \subseteq N^{\cup}$. We distinguish the following cases:

(a) For all $i \in N^{\cup}$, $b_i > c_{im(i)}$.
In this case, we define the allocation:

$$x_i = \begin{cases} b_i - \kappa_1 & \text{if } i \in N^{\cup} \quad \text{is not directly connected to the source} \\ b_i - c_{i\omega} & \text{if } i \in N^{\cup} \quad \text{is directly connected to the source} \\ 0 & \text{if } i \notin N^{\cup} \end{cases}$$

It is clear that $x(N) = v(N)$. Let $T \subseteq N^{\cup}$. From Claim 2, $\mu(T) = v(T) = \pi(T')$, for some sub-coalition $T' \subseteq T$. From Claim 3

$$\mu(T) = \pi(T') = \sum_{i \in T'} b_i - \delta_{T'}\, \kappa_2 - \left(\left|T'\right| - \delta_{T'}\right) \kappa_1 \leq$$

$$\leq \sum_{i \in T'} b_i - \theta_{T'}\, \kappa_2 - \left(\left|T'\right| - \theta_{T'}\right) \kappa_1 = x\left(T'\right) \leq x(T)$$

and the result follows.

(b) There are some agents $i \in N^{\cup}$ such that $b_i \leq c_{im(i)}$.
The agents in an effective coalition such that their revenue is lower or equal to the cost they incur are called *necessary agents*. The idea in this proof is to eliminate these agents by modifying the revenues of the agents connected to them so that the joint net revenue does not change. We proceed as in Example 1:

`[step 1]`: If i^* is a necessary agent, we set $x_{i^*} = 0$, then we remove this agent from $N^{\cup}$ and modify the revenues of the agents that connect to i^* by subtracting the amount $d_{i^*} = \left|b_{i^*} - c_{i\, m(i)}\right|$ equally shared among them; that is: for all $j \in N^{\cup}$ such that $m(j) = i^*$,

$$b'_j = b_j - \frac{d_{i^*}}{k_{i^*}} = b_j - \frac{\left|b_{i^*} - c_{i\, m(i)}\right|}{k_{i^*}}$$

where k_{i^*} denotes the number of agents connecting to i^*. Let $N^{(1)}$ denote the set of non-removed agents.

`[step 2]`: If the new problem has no necessary agents, proceed as in case (*a*) and define the remaining allocations x_i.

`[step 3]`: If the new problem has necessary agents, return to `[step 1]`.

Note that the procedure we follow in `[step 1]` implies that $v(N^{\cup}) = v(N^{(1)})$, hence $x(N) = v(N^{(1)})$.

[3] We call such a simple component "expensive component", whereas the simple components that connect to the source at low cost are called "cheap components".

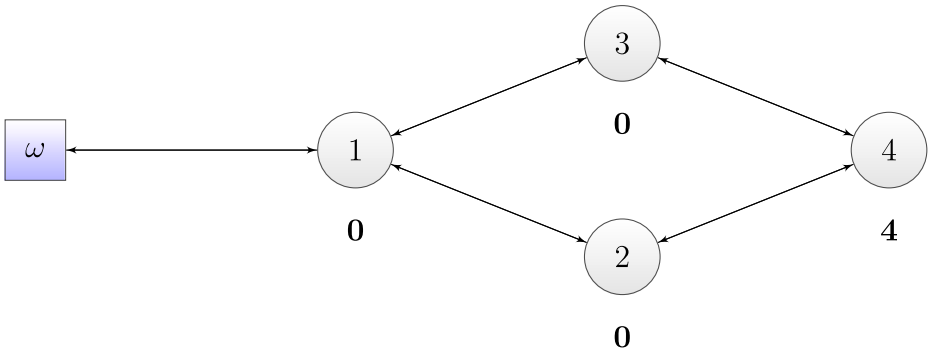


Fig. 4. Elementary cost *mcst* problem with revenues with a non-empty *r-core*. The cost of each link appearing in the picture is $c_{ij} = 1$, whereas the cost of the undrawn links is $c_{ij} = 4$. The boldfaced numbers at the nodes represent the revenues that the corresponding agents obtain.

Now, for any $S \subseteq N^{\cup}$ by reasoning as above we know that $\mu(S) = \upsilon(S) = \pi(S^{(1)})$, $S^{(1)} \subseteq S$. If we apply the algorithm described in `[step 1]` to `[step 3]` to the coalition $S^{(1)}$, we get a sub-coalition of non-necessary agents $S^{(2)} \subseteq S^{(1)}$ such that $\pi(S^{(1)}) = \pi(S^{(2)})$. From case (1) we obtain

$$\mu(S) = \pi(S^{(2)}) \leq x\left(S^{(2)}\right) \leq x(S).$$

Therefore, the *r*-core is non-empty. ■

The following example shows that the assumption in Theorem 2 is not a necessary condition for the non-emptiness of the *r-core* in elementary *mcst* games with revenues.

Example 2. Let us consider the *mcst* problem with revenues represented in Fig. 4.

In this elementary cost *mcst* problem with revenues there are two effective coalitions, namely $E_1 = \{1, 2, 4\}$ and $E_2 = \{1, 3, 4\}$. Moreover, $\upsilon(N) = \pi\left(E_1\right) = \pi\left(E_2\right) = 1$. On the other hand, $N^{\cup} = E_1 \cup E_2 = \{1, 2, 3, 4\} = N$, which is not an effective coalition, since the net revenue is $\pi(\{1, 2, 3, 4\}) = 0$. Despite the failure of the conditions in Estévez-Fernández and Reijnierse (2014) or Theorem 2, the *r-core* is non-empty. Specifically:

$$Core(\upsilon) = \left\{\left(x_1, x_2, x_3, x_4\right) \in \mathbb{R}^3_+ : x_2 = x_3 = 0,\ x_1 + x_4 = 1\right\}.$$

These allocations ensure that the core constraints are satisfied, as agents 2 and 3 receive no payoff, and the net revenue is balanced between agents 1 and 4.

4. Final remarks

We have analyzed *mcst* games with revenues, deriving a necessary condition to ensure the non-emptiness of the *r-core* that resembles the *veto power* condition in simple games. Since cooperation is fundamental in such scenarios, a non-empty *r-core* guarantees the possibility of reaching an agreement.

Additionally, we established a sufficient condition for the non-emptiness of the *r-core* in the specific context of elementary cost *mcst* problems. Our result extend the work of Estévez-Fernández and Reijnierse (2014), which, to our knowledge, is the only prior result addressing the non-emptiness of the *r-core* in this setting. Despite these results, the sufficient conditions do not exhaust all scenarios ensuring the non-emptiness of the *r-core* as Example 2 shows.

In general *mcst* games with revenues, whenever the *r*-core can be empty, Subiza et al. (2025) suggest the use of alternative notions of stability.

CRediT authorship contribution statement

Begoña Subiza: Writing – original draft, Supervision, Methodology, Formal analysis. **José-Manuel Giménez-Gómez:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis. **Josep E. Peris:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis.

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Data availability

No data was used for the research described in the article.

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