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Fitting an Intuitionistic Fuzzy Black-Derman-Toy Model of Yield Curve to Price Interest Rate Derivatives

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Abstract

Since the early 21st century, the Fuzzy Random Option Pricing (FROP) literature has emerged as an active field within fuzzy mathematics. Mainstream contributions model the epistemic uncertainty of parameters governing the stochastic variation of underlying asset prices through Fuzzy Numbers (FNs). Within discrete-time models, virtually all FROP contributions assume the analytical framework of the Cox, Ross, and Rubinstein model (CRR), introducing uncertainty associated with volatility through FNs, with analyses focusing on options on stock and real options. This work falls within this setting, as we extend the nonarbitrage model of the Black, Derman, and Toy (BDT) interest rate term structure to the hypothesis that short-term interest rate volatility is estimated through Intuitionistic Fuzzy Numbers (IFNs), a generalization of FNs that allows the introduction of bipolar uncertainty. As a preliminary step, we propose a methodology to estimate short-term interest rate volatility as an IFN based on transforming probabilistic information into possibilistic information using empirical data and adopting the historical volatility approach. Additionally, we apply the intuitionistic BDT estimate of the yield curve to price options on zero coupon bonds, which is a novelty in the field. We believe that introducing information through IFNs is of particular interest in fixed-income derivative instruments, as they are traded in over-the-counter markets, which are less liquid than exchange markets, and negotiated contracts are usually not standardized.

Keywords: Intuitionistic fuzzy numbers, Probability–possibility transformation, Black-derman-toy model of yield curve, Fuzzy binomial lattices, Options on zero coupon bonds.

1 | Introduction

The publication of the well-known Black-Scholes-Merton (BSM) model for the valuation of European options [1], [2] marked a milestone in financial economics due to its broad implications for financial valuation. The approach used, based on the no-arbitrage argument, can be applied not only to stock option pricing but also to modelling the prices of assets sensitive to interest rates, in the field of capital budgeting, or in the valuation of conventional bank agreements such as loans [3], [4]. It is challenging for continuous-time option

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pricing models to price options that do not have European option structures. This encompasses a wide range of options, such as American options, Bemuda options, and multiple optinalities linked with real options [5]. Thus, a main approach to BSM in discrete-time is the binomial model [6] Cox, Ross and Rubinstein (CRR) and Boyle's trinomial extension [7]. These extensions allow the valuation of a much wider range of optionalities [5].

Conventional option-pricing models assume that the movement of underlying asset prices follows a stochastic process with crisp parameters. However, in most cases, there is uncertainty, vagueness, and/or imprecision in the values. This uncertainty has been captured in multiple studies using tools from fuzzy subset theory, an approach that can be referred to as Fuzzy Random Option Pricing (FROP) [8]. In this stream of fuzzy set literature, statistical parameters describing price fluctuations, such as volatility, are often described using Fuzzy Numbers (FNs), although more nuanced expressions of uncertainty, such as soft sets and Intuitionistic Fuzzy Numbers (IFNs), have been used in some studies [8]. In an FROP setting, the use of an FN $\tilde{\theta}$ to describe a parameter θ whose value is uncertain should be interpreted as an epistemic fuzzy set approximating the real value of the parameter whose membership function is 1 at the most plausible value or set of values [9]. That is, following the definition of epistemic uncertainty by Dubois and Prade [10], the possibility distribution function associated with $\tilde{\theta}$ attempts to roughly capture the knowledge of θ.

FROP has been developed in both continuous-time and discrete-time models. In the first type of model, the most commonly used analytical framework is that of the Brownian geometric movement on which the BSM is based [11–17] or that based on extensions of Brownian-geometric models [18–20]. However, FROP has also modelled the parameters governing option pricing formulas based on jump-diffusion processes [21] or Levy stochastic processes [22–25] with possibility distributions. The development of FROP in discrete time has essentially been based on the CRR binomial model [26–33] and, to a lesser extent, on the trinomial model [7], such as that of [34], [35]. The mainstream FROP literature, both in continuous and discrete time, has been conducted, albeit not exclusively, predominantly in stock options and real options [36]. Other notable applications include life insurance guarantees [23], catastrophe bonds, and forward contracts on energy commodities [25].

Although options theory is widely applied in modelling the yield curve and evaluating contingent assets whose cash flows are sensitive to interest rate fluctuations [4], [5] developments in FROP in this regard are scarce in continuous time and practically nonexistent in discrete time. To the best of our knowledge, we can only highlight the extension to Jamshidian's model [37] in continuous time and the fuzzification of Ho and Lee's model [38], [39]. In all cases, type-I FNs were used to model volatility. The reflections presented in the previous paragraphs motivate this study, which develops the discrete-time term structure model of interest rates by Black, Derman, and Toy (BDT) [40] (hereafter BDT), using a binomial tree similar to CRR, under the assumption that the volatility of short-term interest rates is quantified through an IFN. The developed intuitionistic yield curve was applied to the valuation of European options on zero-coupon bonds.

This study contributes to the FROP in the following ways. First, we model uncertainty using IFNs, which enable the incorporation of bipolar uncertainty. That is, IFNs allow us to represent not only the values that a parameter is likely to take but also those that are considered implausible under any circumstances by the decision maker [41]. While some contributions have applied IFNs in FROP contexts [42–44], their use remains relatively rare. Second, we adapt the BDT binomial yield curve model to incorporate bipolar uncertainty in short-term interest rate volatility. To the best of our knowledge, no prior work has applied IFNs to the BDT framework. Although fuzzy extensions of other interest rate models exist [39], these have relied exclusively on type-I FNs. Thus, our study represents the first attempt to integrate an intuitionistic fuzzy volatility structure into the BDT model and to apply it to the pricing of options on zero-coupon bonds.

Third, we estimate volatility as an IFN using historical data [45]. This is based on interpreting the α-cuts of possibility distributions as nested confidence intervals that include the parameter of interest at varying levels of significance [46]. Since an IFN can be viewed as a variable bounded by upper and lower possibility distributions [10], we build on prior work [47], [48] that constructs fuzzy estimates by nesting empirical

confidence intervals. This methodology has been applied in FROP studies of stock options, real options [48], and to a limited extent in yield curve modeling and bond options [39].

The remainder of this paper is organized as follows. In the next section, we present the groundwork for IFNs used in this study and propose a methodology that allows the fitting of statistical parameters with IFNs from empirical data. We pay special attention to the variance and standard deviation because they are closely linked to the volatility of the subjacent asset in an option-pricing setting. In the third section, we describe the term structure BDT model and how to apply it to price interest-rate-contingent assets, with a special focus on European call and put options on zero-coupon bonds. Subsequently, we developed our intuitionistic fuzzy version of the BDT model and applied it to the valuation of the same options. The developments are developed using empirical data from interest rates traded in the European fixed-income market on securities maturing for up to five years and rated AAA, than can be assimilated to those credit-risk free.

2 | Fitting Intuitionistic Fuzzy Numbers from Empirical Data and Intuitionistic Fuzzy Number Arithmetics

2.1 | Fitting Staticatisl Parameters with Fuzzy Numbers

Definition 1 ([49]). A fuzzy set is defined on X, $\widetilde{A}$: $\widetilde{A} = \{\langle x, \mu_A(x)\rangle, x \in X\}$, where $\mu_A: X \longrightarrow [0,1]$. Conversely, $\widetilde{A}$ is captured via α-cuts, A_α: $A_\alpha = \{x|\mu_A(x) \geq \alpha, \alpha\in(0,1]\}$.

Definition 2. A FNs, $\widetilde{A}$, is defined in $\mathbb{R}$ that accomplishes:

I. $\exists x|\mu_A(x) = 1$.

II. For all $x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1, \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(\mu_A(x_1), \mu_A(x_2)\right)$.

Remark 1. Consequently, the α-cuts of $\widetilde{A}$ and A_α are the confidence intervals [50].

$$\underline{B'}_\beta = f\left(\underline{A'_{(1)_\beta}}, \underline{A'_{(2)_\beta}}, \dots, \underline{A'_{(m)_\beta}}, \overline{A'_{(m+1)_\beta}}, \dots, \overline{A'_{(n)_\beta}}\right). \quad \textbf{(1)}$$

$$A_\alpha = \{x|\mu_A(x) \geq \alpha, \alpha\in(0,1]\} = \left[\underline{A}_\alpha, \overline{A}_\alpha\right],$$

where $\underline{A}_\alpha$ and $\overline{A}_\alpha$ are increasing and decreasing functions of α, respectively. FN $\widetilde{A}$ is often referred to as a possibility distribution. Fuzzy set theory applies FNs to represent uncertain values [50] and, similar to Random Variable (RV), FNs quantify an unknown parameter. The coherent transformation of RV to possible distributions has been widely analyzed in the literature [51], [52].

Definition 3. Suppose for a continuous RV a set of intervals A_α such that $P(x \in A_\alpha) \geq 1-\alpha$, where P(·) stands for probability. The α-cuts of the equivalent fuzzy representation $\widetilde{A}$ are [46].

$$A_\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right] = \left[\left\{x|P(\mathbf{A} \leq x) = \frac{\alpha}{2}\right\}, \left\{x|P(\mathbf{A} \leq x) = 1 - \frac{\alpha}{2}\right\}\right]. \quad \textbf{(2)}$$

Remark 2. Consequently, the membership function of $\widetilde{A}$ equivalent to A is $\mu_A(x) = \sup\{\alpha|x\epsilon A_\alpha\}$. From this conceptualization of the link between rvs and FNs, the literature has proposed the estimation of statistical parameters, such as the mean or variance, as possibility distributions. These FNs are obtained by overlaying the sample confidence intervals of the parameter of interest from the lowest to the highest level of significance [47], [53–56]. Therefore, we define:

Definition 4. By following *Definition 3*, let us have [47]:

I. A sample of a set of iid rvs with an associated unknown parameter θ that obtained estimates with a significance level (1-α)%, $\left[\underline{\theta}(\alpha), \overline{\theta}(\alpha)\right]$.

II. A monotonic function $h(\gamma, \alpha)$: $(0,1] \rightarrow \left[\frac{\gamma}{2}, 0.5\right], \gamma\epsilon(0,1)$. It links the degree of significance of the statistical confidence interval to a probabilistic measure.

Then, from $[\underline{\theta}(\alpha), \overline{\theta}(\alpha)]$, we can build a fuzzy equivalent for θ, ${}^{\gamma}\tilde{\theta}$ whose α-cuts are

$$ {}^{\gamma}\theta_{\alpha} = \left[\underline{{}^{\gamma}\theta_{\alpha}}, \overline{{}^{\gamma}\theta_{\alpha}} \right] = [\underline{\theta}(2h(\gamma,\alpha)), \overline{\theta}(2h(\gamma,\alpha))]. \quad (3) $$

Remark 3. In this approach, the parameter γ sets the significance level of the confidence interval that serves as the support or 0-cut of ${}^{\gamma}\tilde{\theta}$, onto which confidence intervals with significance levels $\alpha\epsilon(0,1]$ are overlaid. Thus, as γ decreased, the inclusivity of ${}^{\gamma}\tilde{\theta}$ increased. The fitting procedure employs confidence intervals at multiple significance levels to construct nested α-cuts of the possibility distribution. This approach is inspired by epistemic fuzzy modeling, where statistical intervals serve as approximations of knowledge bounds, rather than representations of frequentist probabilities [51–55].

The determination of γ to state all the possible values to take into account in ${}^{\gamma}\tilde{\theta}$ is interpreted in this paper from the perspective of the 95% and 99.7% rules, which are used as rules of thumb in numerous settings, such as finance and business [57]. A value of 95% (i.e., a significance level of 5%) includes common scenarios; thus, they include those that are reasonably but not completely extreme. However, the consideration of probabilities γ near 0, like 0.3% (99.7 rule), practically assumes all possible scenarios. So, the choice of 95% and 99.7% confidence levels is not arbitrary, but rather widely adopted in financial modeling practice. These levels correspond to the empirical "two sigma" (95%) and "three sigma" (99.7%) rules commonly used to delineate reasonably expected versus extreme scenarios under the assumption of normality. Their relevance is particularly evident in applications such as risk management, derivative pricing, and regulatory frameworks like Value-at-Risk (VaR), where 95% often serves as a threshold for risk tolerance and probabilities beyond 99% are used for stress-testing and tail risk assessment [58].

Remark 4. In this paper, for $h(\gamma,\alpha)$, the following linear function was used. Its development is justified in [47]:

$$ h(\gamma,\alpha) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\alpha + \frac{\gamma}{2}. \quad (4) $$

The linear transformation function in *Eq. (4)* is chosen for its simplicity, monotonicity, and consistency with the nested structure of confidence intervals and justified in [47]. While other mappings could be employed, the linear form preserves interpretability and computational tractability.

Definition 5. Let be a set of n observations with an empirical variance s^2. For variance σ^2, the FNs approximation ${}^{\gamma}\tilde{\sigma}^2$ [47] assumes that the variance σ^2 is estimated from a sample of n independent observations of a normally distributed variable. The classical $1-\gamma$ confidence interval is given by $\left[\frac{(n-1)s^2}{\chi^2_{n-1;1-\frac{\gamma}{2}}}, \frac{(n-1)s^2}{\chi^2_{n-1;\frac{\gamma}{2}}}\right]$. Given that statistical confidence intervals can be interpreted as possibilitic confidence intervals, from *Eqs. (3)* and *(4)*. [47] states:

$$ \mu_{{}^{\gamma}\sigma^2}(x) = \begin{cases} \dfrac{2-\gamma}{1-\gamma} - \dfrac{2}{1-\gamma} F\left(\dfrac{(n-1)s^2}{x}\right) & \dfrac{(n-1)s^2}{\chi^2_{n-1;\frac{2-\gamma}{2}}} \le x \le \dfrac{(n-1)s^2}{M} \\ \dfrac{2}{1-\gamma} F\left(\dfrac{(n-1)s^2}{x}\right) - \dfrac{\gamma}{1-\gamma} & \dfrac{(n-1)s^2}{M} \le x \le \dfrac{(n-1)s^2}{\chi^2_{n-1;\frac{\gamma}{2}}}, \end{cases} \quad (4a) $$

being $F(\cdot)$ is distribution function of a chi-square n with $n-1$ degrees of freedom, $\chi^2_{n-1;(\cdot)}$ its counter-image or a probability level $(\cdot)$, and M is 50th percentile of the chi-square rv. Thus, ${}^{\gamma}\tilde{\sigma}^2$ is represented by

$$ {}^{\gamma}\sigma^2_{\alpha} = \left[\underline{{}^{\gamma}\sigma^2_{\alpha}}, \overline{{}^{\gamma}\sigma^2_{\alpha}} \right] = \left[\frac{(n-1)s^2}{\chi^2_{n-1;\frac{2-\gamma}{2}-\left(\frac{1}{2}-\frac{\gamma}{2}\right)\alpha}}, \frac{(n-1)s^2}{\chi^2_{n-1;\left(\frac{1}{2}-\frac{\gamma}{2}\right)\alpha+\frac{\gamma}{2}}} \right]. \quad (4b) $$

Remark 5. The possibility distribution function of the standard deviation, σ , ${}^{\gamma}\tilde{\sigma}$, comes from $\mu_{{}^{\gamma}\sigma}(x) = \mu_{{}^{\gamma}\sigma^2}(x^2)$, thus, the α-cuts ${}^{\gamma}\sigma_{\alpha}$ are:

$$ {}^{\gamma}\sigma_{\alpha} = \left[\underline{{}^{\gamma}\sigma_{\alpha}}, \overline{{}^{\gamma}\sigma_{\alpha}}\right] = \left[\sqrt{\underline{{}^{\gamma}\sigma_{\alpha}^2}}, \sqrt{\overline{{}^{\gamma}\sigma_{\alpha}^2}}\right]. \quad \textbf{(5)} $$

2.2 | Using Intutitionistic Fuzzy Numbers to Model Statistical Parameters

Definition 6. The Intuitionistic Fuzzy Set (IFS) $\tilde{A}^I$ in a set X is $\tilde{A}^I = \{\langle x, \mu_A(x), v_A(x)\rangle, x \in X\}$.Thus, $\mu_A: X \longrightarrow [0,1]$ quantifies the membership degree and $v_A: X \longrightarrow [0,1]$, nonmembership. The following relation holds: $0 \leq \mu_A(x) + v_A(x) \leq 1$ [59].

Remark 6. Hesitancy degree in $\tilde{A}^I$, $h_A(x)$, is: $h_A(x) = 1 - \mu_A(x) - v_A(x)$. Notice that an IFS extends fuzzy sets since when $h_A(x) = 0$ for all x, $\tilde{A}^I$ turns into $\tilde{A}$.

Definition 7. The $\langle\alpha, \beta\rangle$-cut of $\tilde{A}^I$, $A_{\langle\alpha,\beta\rangle}$ is [59]:

$$ A_{\langle\alpha,\beta\rangle} = \{x|\mu_A(x) \geq \alpha, v_A(x) \leq \beta, 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in [0,1]\}. \quad \textbf{(6a)} $$

Therefore, $A_{\langle\alpha,\beta\rangle}$ is split as $A_{\alpha} = \{x|\mu_A(x) \geq \alpha\}$, and $A'_{\beta} = \{x|v_A(x) \leq \beta\}$, such that [59]:

$$ A_{\langle\alpha,\beta\rangle} = \langle A_{\alpha} = \{x|\mu_A(x) \geq \alpha\}, A'_{\beta} = \{x|v_A(x) \leq \beta\}, 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in [0,1]\rangle. \quad \textbf{(6b)} $$

Definition 8. An IFN is an IFS defined on $\mathbb{R}$ such that [60].

$$ \exists x|\mu_A(x) = 1 \Rightarrow v_A(x) = h_A(x) = 0. $$
$$ \text{for all } x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1, \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(\mu_A(x_1), \mu_A(x_2)\right), $$

And $v_A(x)$ are concave; that is,

$$ \text{for all } x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1, v_A(\lambda x_1 + (1-\lambda)x_2) \leq \max\left(v_A(x_1), v_A(x_2)\right). $$

Remark 7. The $\langle\alpha, \beta\rangle$-cuts of $\tilde{A}^I$, $A_{\langle\alpha,\beta\rangle}$, can be decoupled as

$$ A_{\alpha} = \{x|\mu_A(x) \geq \alpha\} = \left[\underline{A_{\alpha}}, \overline{A_{\alpha}}\right] \text{ and } A'_{\beta} = \{v_A(x) \leq \beta\} = \left[\underline{A'_{\beta}}, \overline{A'_{\beta}}\right], 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in (0,1), $$

Remark 8. Thus, $A_{\langle\alpha,\beta\rangle}$ can be represented as

$$ A_{\langle\alpha,\beta\rangle} = \langle A_{\alpha} = \left[\underline{A_{\alpha}}, \overline{A_{\alpha}}\right], A'_{\beta} = \left[\underline{A'_{\beta}}, \overline{A'_{\beta}}\right], 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in (0,1)\rangle, \quad \textbf{(6c)} $$

where $\underline{A_{\alpha}}$ and $\overline{A'_{\beta}}$ increase with respect to α and β, respectively. Similarly, $\overline{A_{\alpha}}$ and $\underline{A'_{\beta}}$ decreased with respect to these variables.

Remark 9. $\mu_A(x)$ is the lower membership degree of A, and $\mu_A^*(x) = 1 - v_A(x)$ is the upper membership degree. Thus, the level set of the possibility distribution $\tilde{A}^*$, ${A^*}_{\alpha}$ and the β-cut in *Eq. (6c)*, A'_{β}, holds when $\beta = 1 - \alpha$, $A'_{1-\alpha} = {A^*}_{\alpha}$, that is,

So, $\mu_A^*(x)$ and $\mu_A(x)$ can be understood as bipolar valuations of the possibility that A is x [10]. Therefore, $\mu_A^*(x)$ can be interpreted as the potential membership degree and $\mu_A(x)$ is the actual possibility. *Fig. 1* shows a triangular IFN.

$$ \left[\underline{A'_{1-\alpha}}, \overline{A'_{1-\alpha}}\right] = \left[\underline{{A^*}_{\alpha}}, \overline{{A^*}_{\alpha}}\right]. \quad \textbf{(7)} $$

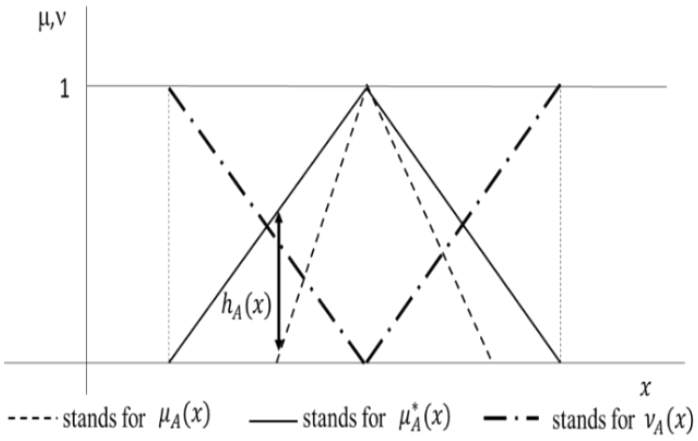


Fig. 1. Shape of a triangular IFN.

Based on this conceptualization of the connection between RVs and possibility distributions, several authors have proposed the estimation of statistical parameters such as the mean or variance as possibility distributions. These FNs are obtained by overlaying the sample confidence intervals of the parameter of interest from the lowest to the highest level of significance [53–56]. Therefore, we define:

Definition 9. As in *Definition 4*, we consider a sample generated by an rv ruled by an unknown parameter θ and, which allows us to estimate θ by means of an interval $\left[\underline{\theta}(\alpha), \overline{\theta}(\alpha)\right]$ with significance (1-α)%. By following [45], we fit and IFN $\tilde{\theta}^I$ by adjusting its lower membership function by stating in *Eq. (2)* γ, $({}^{\gamma}\tilde{\theta})$, that is, $\mu_\theta(x) = \mu_{{}^{\gamma}\theta}(x)$ and its upper membership degree by means $\gamma^* \leq \gamma$ in *Eq. (2)*, $({}^{\gamma^*}\tilde{\theta})$, i.e., $\mu_\theta^*(x) = \mu_{{}^{\gamma^*}\theta}(x)$. Therefore, for $\tilde{\theta}^I$ we can state that:

$$\mu_\theta(x) = \mu_{{}^{\gamma}\theta}(x), \mu_\theta^*(x) = \mu_{{}^{\gamma^*}\theta}(x), \upsilon_\theta(x) = 1 - \mu_{{}^{\gamma^*}\theta}(x), \tag{8a}$$

Therefore, $\theta_{\langle\alpha,\beta\rangle} = \langle\theta_\alpha = \left[\underline{\theta_\alpha}, \overline{\theta_\alpha}\right], \theta'_\beta = \left[\underline{\theta'_\beta}, \overline{\theta'_\beta}\right], 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in (0,1)\rangle$, where:

$$\theta_\alpha = {}^{\gamma}\theta_\alpha = \left[\underline{{}^{\gamma}\theta_\alpha}, \overline{{}^{\gamma}\theta_\alpha}\right] = \left[\underline{\theta}(2h(\gamma,\alpha)), \overline{\theta}(2h(\gamma,\alpha))\right], \tag{8b}$$

$$\theta'_\beta = {}^{\gamma^*}\theta_{1-\beta} = \left[\underline{{}^{\gamma^*}\theta_{1-\beta}}, \overline{{}^{\gamma^*}\theta_{1-\beta}}\right] = \left[\underline{\theta}(2h(\gamma^*, 1-\beta)), \overline{\theta}(2h(\gamma^*, 1-\beta))\right] \tag{8c}$$

Definition 10. Suppose that the empirical variance s^2 is obtained from a sample size n. *Definition 5* allows fitting and IFN variance $\tilde{\sigma}^{I^2}$ from ${}^{\gamma}\tilde{\sigma}^2$ and ${}^{\gamma^*}\tilde{\sigma}^2$. From *Eq. (8a)*, the possibilistic fit $\tilde{\sigma}^{I^2} = \{\langle x, \mu_{\sigma^2}(x), \nu_{\sigma^2}(x)\rangle, x \in X\}$, is

$$\mu_{\sigma^2}(x) = \mu_{{}^{\gamma}\sigma^2}(x), \mu_{\sigma^2}^*(x) = \mu_{{}^{\gamma^*}\sigma^2}(x) \text{ and } \nu_{\sigma^2}(x) = 1 - \mu_{\sigma^2}^*(x). \tag{9a}$$

Therefore, $\sigma^2_{\langle\alpha,\beta\rangle}$ can be denoted as, $\sigma^2_{\langle\alpha,\beta\rangle} = \langle\sigma^2_\alpha = \left[\underline{\sigma^2_\alpha}, \overline{\sigma^2_\alpha}\right], \sigma^{2\prime}_\beta = \left[\underline{\sigma^{2\prime}_\beta}, \overline{\sigma^{2\prime}_\beta}\right], 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in (0,1)\rangle$ where by using *Eqs. (4b)* and *(8b)*:

$$\sigma^2_\alpha = {}^{\gamma}\sigma^2_\alpha = \left[\underline{{}^{\gamma}\sigma^2_\alpha}, \overline{{}^{\gamma}\sigma^2_\alpha}\right] = \left[\frac{(n-1)s^2}{\chi^2_{n-1;\frac{2-\gamma}{2}-\left(\frac{1}{2}-\frac{\gamma}{2}\right)\alpha}}, \frac{(n-1)s^2}{\chi^2_{n-1;\left(\frac{1}{2}-\frac{\gamma}{2}\right)\alpha+\frac{\gamma}{2}}}\right]. \tag{9b}$$

From *Eqs. (4b)* and *(8c)*,

$$\sigma_{\beta}^{2\prime} = \left[\underline{\gamma^{*}\sigma_{1-\beta}^{2}}, \overline{\gamma^{*}\sigma_{1-\beta}^{2}}\right] = \left[\frac{(n-1)s^{2}}{\chi^{2}_{n-1;\frac{2-\gamma^{*}}{2}-\left(\frac{1}{2}-\frac{\gamma^{*}}{2}\right)(1-\beta)}}, \frac{(n-1)s^{2}}{\chi^{2}_{n-1;\left(\frac{1}{2}-\frac{\gamma^{*}}{2}\right)(1-\beta)+\frac{\gamma^{*}}{2}}}\right]. \quad \textbf{(9c)}$$

Remark 10. Thus, we can adjust and IFNs $\tilde{\sigma}^{I}$ to σ, t by stating $\mu_{\sigma}(x) = \mu_{\sigma^2}(x^2)$ and $\upsilon_{\sigma}(x) = \upsilon_{\sigma^2}(x^2)$. Thus, the $\langle\alpha,\beta\rangle$-cuts $\sigma_{\langle\alpha,\beta\rangle} = \langle\sigma_{\alpha} = \left[\underline{\sigma_{\alpha}}, \overline{\sigma_{\alpha}}\right], \sigma'_{\beta} = \left[\underline{\sigma'_{\beta}}, \overline{\sigma'_{\beta}}\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in (0,1)\rangle$ are:

$$^{\gamma}\sigma_{\alpha} = \left[\sqrt{\underline{\gamma\sigma_{\alpha}^{2}}}, \sqrt{\overline{\gamma\sigma_{\alpha}^{2}}}\right] \text{ and } \sigma'_{\beta} = \left[\sqrt{\underline{\sigma_{\beta}^{2\prime}}}, \sqrt{\overline{\sigma_{\beta}^{2\prime}}}\right]. \quad \textbf{(9d)}$$

Numerical application

The numerical applications in this study rely on the term structure of interest rates for European public bonds with the highest credit rating AAA. These data are available from the European Central Bank website [61], and consist of daily observations from May 30, 2022, to May 30, 2023 (n = 258). The annualized volatility was estimated using the standard deviation of daily rates over this period, a widely accepted approach in both financial practice and academic literature (see, e.g., [8], [58]), especially when option-implied volatilities are not available. It is so called historical volatility. Although the one-year window may include macroeconomic shocks, its use in this paper is intended for illustrative purposes, not as a predictive tool. The aim is to construct an intuitionistic fuzzy representation of volatility based on actual market data. Future research could consider longer time horizons or adjustments for exceptional market conditions to assess the robustness of this methodology. *Fig. 2* shows the values of the one-year interest rate and its variation during this period.

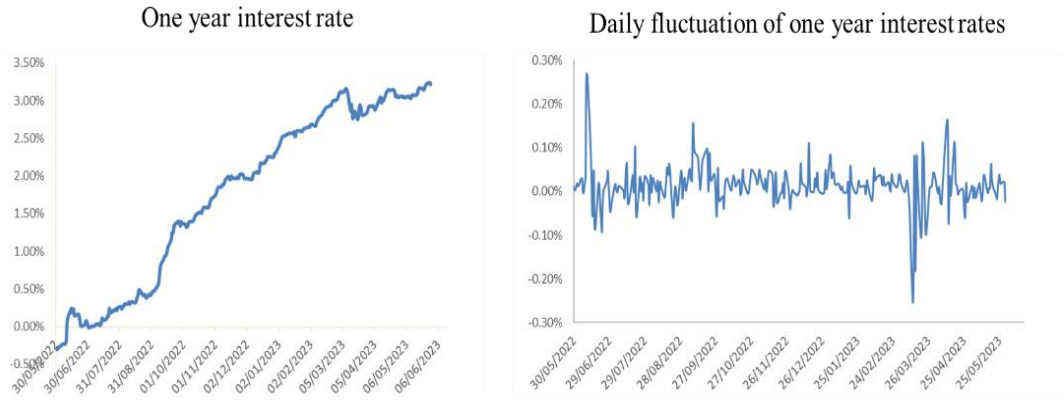


Fig. 2. Evolution of one-year interest rates for securities rated as AAA in the European Union markets from May 30, 2022, to May 30, 2023.

The annualized volatility of the interest rate is 0.714% with n=258 observations. For a significance probability γ=5% (i.e., using the 95% rule), we obtain a distribution function for the annualized volatility $^{0.05}\tilde{\sigma}$, whose shape starts from *Eq. (4a)* and *Remark 5*:

$$\mu_{0.05\sigma}(x) = \begin{cases} \dfrac{1.95}{0.95} - \dfrac{2}{0.95}F\left(\dfrac{257\cdot 0.00714^{2}}{x^{2}}\right) & 0.00657^{2} \le x \le 0.00715^{2}, \\ \dfrac{2}{0.95}F\left(\dfrac{257\cdot 0.00714^{2}}{x^{2}}\right) - \dfrac{0.05}{0.95} & 0.00715^{2} \le x \le 0.00783^{2}. \end{cases}$$

Therefore, the α-cuts *Eq. (4b)* are:

$$ {}^{0.05}\sigma_\alpha = \left[\underline{{}^{0.05}\sigma_\alpha}, \overline{{}^{0.05}\sigma_\alpha^2}\right] = \left[\sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;0.975-\frac{0.95}{2}\alpha}}}\%, \sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;\frac{0.95}{2}\alpha+0.025}}}\%\right]. $$

We now fit an IFN intuitinistic $\tilde{\sigma}^I$ based on ${}^{0.05}\tilde{\sigma}$, which we can consider as the lower possibility distribution, and adjust an upper membership function whose support is fitted with the 99.7% rule. That is, it contains the values included in the confidence interval values whose significance level is $\gamma *= 0.3\%$, ${}^{0.003}\tilde{\sigma}$. Thus, $\mu_\sigma(x) = \mu_{{}^{0.05}\sigma}(x)$:

$$ \mu_\sigma(x) = \begin{cases} \frac{1.95}{0.95} - \frac{2}{0.95}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) & 0.00657^2 \le x \le 0.00715^2, \\ \frac{2}{0.95}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) - \frac{0.05}{0.95} & 0.00715^2 \le x \le 0.00783^2. \end{cases} $$

From *Eq. (4a)*, $\mu^*_\sigma(x) = \mu_{{}^{0.003}\sigma}(x)$:

$$ \mu^*_\sigma(x) = \begin{cases} \frac{1.997}{0.997} - \frac{2}{0.997}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) & 0.00630^2 \le x \le 0.00715^2, \\ \frac{2}{0.997}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) - \frac{0.003}{0.997} & 0.00715^2 \le x \le 0.00822^2. \end{cases} $$

Thus, $\nu_\sigma(x) = 1 - \mu^*_\sigma(x)$:

$$ \nu_\sigma(x) = \begin{cases} \frac{2}{0.997}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) - \frac{1}{0.997} - & 0.00630^2 \le x \le 0.00715^2, \\ \frac{1}{0.997} - \frac{2}{0.997}F\left(\frac{257\cdot 0.00714^2}{x^2}\right) & 0.00715^2 \le x \le 0.00822^2. \end{cases} $$

Similarly, $\sigma_{\langle\alpha,\beta\rangle}$ is defined as

$$ \sigma_\alpha = {}^{\gamma}\sigma_\alpha = \left[\underline{{}^{\gamma}\sigma_\alpha}, \overline{{}^{\gamma}\sigma_\alpha}\right] = \left[\sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;0.975-\frac{0.95}{2}\alpha}}}\%, \sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;\frac{0.95}{2}\alpha+0.025}}}\%\right], $$

and

$$ \sigma'_\beta = \left[\underline{{}^{\gamma^*}\sigma_{1-\beta}}, \overline{{}^{\gamma^*}\sigma_{1-\beta}}\right] = \left[\sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;0.9985-\frac{0.997}{2}(1-\beta)}}}\%, \sqrt{\frac{257\cdot 0.714^2}{\chi^2_{257;\frac{0.997}{2}(1-\beta)+0.0015}}}\%\right]. $$

Fig. 3 displays the possibilistic representation of the annaulized standard deviation of the one-year interest rate, ${}^{0.05}\tilde{\sigma}$ and ${}^{0.003}\tilde{\sigma}$, and the IFN developed above.

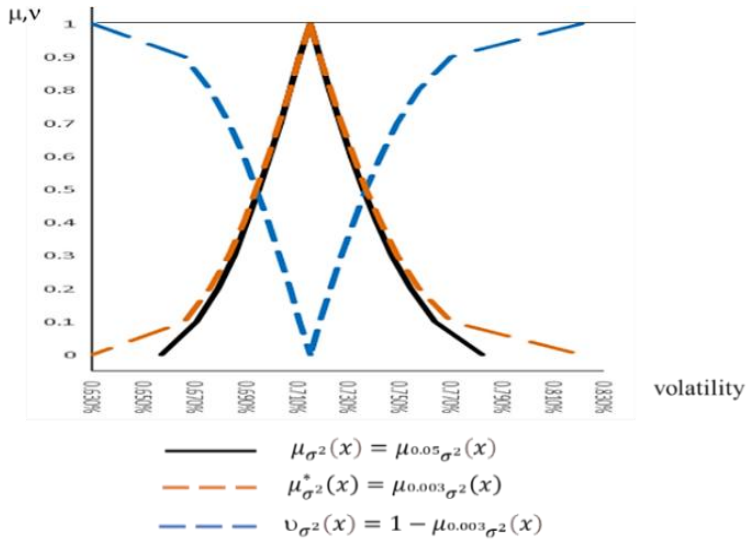


Fig. 3. FN and IFN representation of one-year interest rates for AAA-rated issuances in European Union fixed income markets on May 30, 2023.

Eq. (4a) follows a standard approach for estimating fuzzy (and intuitionistic fuzzy) representations of variance using the chi-square distribution, which relies on the assumption of normally distributed data. This assumption is commonly adopted in the literature (e.g., [14], [47], [48]) for constructing fuzzy confidence intervals around variance estimates. Moreover, given the relatively large number of observations in our sample (n = 258), this approximation is generally considered robust. According to the Central Limit Theorem, even when the underlying distribution is not strictly normal, the sampling distribution of the variance estimator tends to approximate that of a normal RV as the sample size increases. Alternatively, a bootstrapping procedure within the context of FNs, as suggested in [37], could be employed by adapting *Eq. (2)* and applying the 95% and 99.7% rules to construct IFNs.

2.3 | Intuitionistic Fuzzy Number Arithmetic

In [62–64], the compatibility of Zadeh's [65] extension principle with the evaluation of functions on α-cuts was extended when the arguments of the function were IFNs. Let be a continuous and differentiable function $y = f(x_1, x_2, \dots, x_n)$, such that the values of the input variables are given as the means of IFNs $\widetilde{A}^I_{(i)}, i = 1,2, \dots, n$. It generates an IFN $\widetilde{B}^I$, $\widetilde{B}^I = f\left(\widetilde{A}^I_{(1)}, \widetilde{A}^I_{(2)}, \dots, \widetilde{A}^I_{(n)}\right)$. Thus, the membership and nonmembership functions of $\widetilde{B}^I$ are:

$$\mu_B(y) = \max_{y=f(x_1,x_2,\dots,x_n)} \min\left\{\mu_{A_{(1)}}(x_1), \mu_{A_{(2)}}(x_2), \dots, \mu_{A_{(n)}}(x_n)\right\}, \quad \textbf{(10a)}$$

$$v_B(y) = \min_{y=f(x_1,x_2,\dots,x_n)} \max\left\{v_{A_{(1)}}(x_1), v_{A_{(2)}}(x_2), \dots, v_{A_{(n)}}(x_n)\right\}. \quad \textbf{(10b)}$$

Therefore, if $\widetilde{A}^I_{(i)}$ are FNs $\widetilde{A}_{(i)}$, , $i = 1,2, \dots, n$, $\widetilde{B}$ is also an FN with only $\mu_B(y)$ in *Eq. (10a)*. To calculate $B_{\langle\alpha,\beta\rangle}$ from $A_{(i)\langle\alpha,\beta\rangle}$, $i = 1,2, \dots, n$, we obtain:

$$B_{\langle\alpha,\beta\rangle} = f\left(A_{(1)\langle\alpha,\beta\rangle}, A_{(2)\langle\alpha,\beta\rangle}, \dots, A_{(n)\langle\alpha,\beta\rangle}\right). \quad \textbf{(11a)}$$

Thus, given that f is assumed to be continuous, $B_{\langle\alpha,\beta\rangle}$ can be represented as

$$B_{\langle\alpha,\beta\rangle} = \langle B_\alpha = \left[\underline{B_\alpha}, \overline{B_\alpha}\right], B'_\beta = \left[\underline{B'_\beta}, \overline{B'_\beta}\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in [0,1] \rangle,$$

where:

$$\underline{B_\alpha} = \inf\left\{y | y = f(x_1, \dots, x_n),\ x_i \in A_{(i)\alpha}\right\}, \overline{B_\alpha} = \sup\left\{y | y = f(x_1, \dots, x_n),\ x_i \in A_{(i)\alpha}\right\}, \quad \textbf{(11b)}$$

$$\underline{B'_\beta} = \inf\left\{y|y = f(x_1, \dots, x_n),\ x_i \in A'_{(i)_\beta}\right\}, \overline{B'_\beta} = \sup\left\{y|y = f(x_1, \dots, x_n),\ x_i \in A'_{(i)_\beta}\right\}. \quad \textbf{(11c)}$$

Following [65], [66] when f monotonically increases with respect to $x_i, i = 1,2, \dots m$ and monotonically decreases in $x_i, i = m+1, m+2, \dots, n$, $m \leq n$, $B_\alpha = \left[\underline{B_\alpha}, \overline{B_\alpha}\right]$ is:

$$\underline{B_\alpha} = f\left(\underline{A_{(1)_\alpha}}, \underline{A_{(2)_\alpha}}, \dots, \underline{A_{(m)_\alpha}}, \overline{A_{(m+1)_\alpha}}, \dots, \overline{A_{(n)_\alpha}}\right). \quad \textbf{(11d)}$$

$$\overline{B_\alpha} = f\left(\overline{A_{(1)_\alpha}}, \overline{A_{(2)_\alpha}}, \dots, \overline{A_{(m)_\alpha}}, \underline{A_{(m+1)_\alpha}}, \dots, \underline{A_{(n)_\alpha}}\right). \quad \textbf{(11e)}$$

So, for $B'_\beta = \left[\underline{B'_\beta}, \overline{B'_\beta}\right]$ we state:

$$\underline{B'_\beta} = f\left(\underline{A'_{(1)_\beta}}, \underline{A'_{(2)_\beta}}, \dots, \underline{A'_{(m)_\beta}}, \overline{A'_{(m+1)_\beta}}, \dots, \overline{A'_{(n)_\beta}}\right). \quad \textbf{(11f)}$$

$$\overline{B'_\beta} = f\left(\overline{A'_{(1)_\beta}}, \overline{A'_{(2)_\beta}}, \dots, \overline{A'_{(m)_\beta}}, \underline{A_{(m+1)_\beta}}, \dots, \underline{A_{(n)_\beta}}\right). \quad \textbf{(11g)}$$

If we evaluate $f(\cdot)$ with $\widetilde{A}_{(i)}$, i=1,2,…,n, the α-cuts are $B_\alpha = f\left(A_{(1)_\alpha}, \dots, A_{(n)_\alpha}\right)$ and are fitted by using the general formulation *Eq. (11b)*, and *Eq. (11d)* and *Eq. (11e)* when $f(\cdot)$ is a monotonic function. The propagation of uncertainty through real-valued functions of IFNs can be theoretically justified using an extension of Zadeh's [49], extension principle and applied in the evaluation of functions with fuzzy variables [65], [66]. This explains why under continuity and monotonicity conditions, the α- and β-cuts of the resulting IFN can be computed by evaluating the function over the corresponding cuts of the input variables [63], [64], [68].

Eqs. (11d)–(11g) are derived under the assumption that the function $f(x_1, x_2, \dots, x_n)$ is continuous and piecewise monotonic, which ensures that the endpoints of the IFN output correspond to extrema over the combinations of endpoints of the inputs. This approach is consistent with previous literature [66], [67] and extends the interval arithmetic developed for standard FNs [63], [64]. To maintain rigor, all IFNs are treated via their ⟨α,β⟩-cuts, and the bounds are obtained through interval evaluations. This process ensures the output IFN retains the necessary properties of convexity in the membership and non-membership dimensions, preserving interpretability and consistency within the IFN framework.

3 | Pricing Options on Bonds with the Black-Derman-Toy Model of Temporal Structure of Interest Rates

3.1 | Representing the Dynamics of the Term Structure with the Black-Derman-Toy Model

The use of recombinant binomial trees to capture the evolution of short-yield rates, as shown in *Fig. 4*, is common in the valuation of yield-rate-sensitive derivative instruments [69]. This framework underlies yield curve models based on the no-arbitrage hypothesis, such as that BDT [40], which is the subject of this study. In *Fig. 4*, r_{ij} denotes the short-term interest rate in the jth state of the ith moment, such that j=0,1,…,i. Similarly, the risk-neutral probability of a down move in interest is symbolized as p, which in binomial interest rate models is commonly considered $p = \frac{1}{2}$. Thus, adjusting the temporal structure of binomial interest rates consists of determining interest rates r_{ij} in the binomial tree compatible with the observed yield curve and ensuring that risk-neutral up and down movements are equiprobable [70].

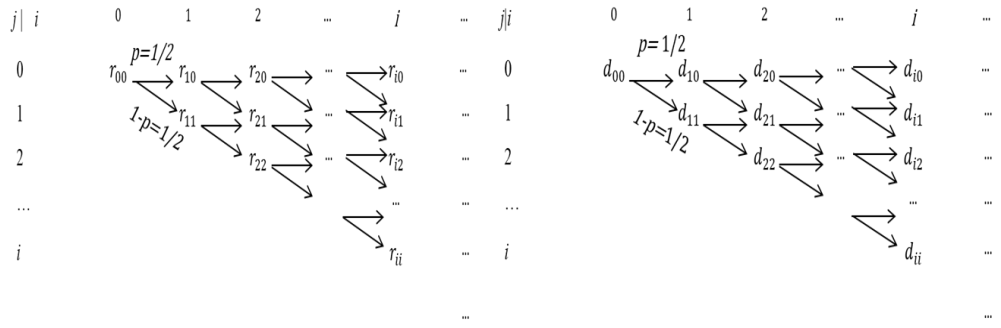


Fig. 4. Short-term yield rate evolution in a binomial lattice and prices of associated Arrow–Debreu assets.

The price of the Arrow-Debreu Assets (ADA) in *Fig. 4* is obtained from the evolution of interest rates. Therefore, d_{ij} stands for the value of an ADA that provides one Monetary Unit (Mu) in the jth state of the ith period. By naming the period of binomial jumps as h, the value d_{ij} is split from $d_{i-1,j-1}$ and $d_{i-1,j}$:

$$d_{ij} = \frac{1}{2}\left(d_{i-1,j-1}e^{-r_{i-1,j-1}h} + d_{i-1,j}e^{-r_{i-1,j}h}\right). \quad (12)$$

In a complete market, the price of an asset that only provides cash flows at maturity, ith years, and with a cash flow in the jth state c_{ij} is:

$$\Pi_i = \sum_{j=0}^{i} c_{ij} \cdot d_{ij}. \quad (13)$$

Therefore, if we denote P_i as a zero-coupon asset that matures in the ith period, then

$$P_i = \sum_{j=0}^{i} d_{ij}. \quad (14)$$

Two versions of the BDT model exist. In its original conceptualization, the volatility of interest rates is variable and, for its implementation, requires using not only a temporal structure of zero coupon rates (the internal rates of return of P_i, i=0,1,…,n), but also the temporal structure of volatilities [40]. A relevant problem with this conceptualization is that volatility must decrease over time, which is not necessarily true in practice [71]. The second conceptualization considers volatility to be a constant parameter related to short-term interest rates, and is the most useful version in practice [71]. Thus, we restrict our development to this second hypothesis, which implies a linear constant-variance Brownian process. Thus, the short-discount rate at t, r_t:

$$d\ln r_t = \theta(t)dt + \sigma dW_t, \quad (15)$$

where $\theta(t)$ is the linear tendency of the logarithm of the interest rate at t, σ annualized volatility is, and dW_t is the normal rv with mean 0 and standard deviation dt. The binomial counterpart of *Eq. (15)* is from [68]:

$$\ln r_{i,j} = \ln r_{i-1,j} + \theta_i h - \sigma\sqrt{h}, \text{and } \ln r_{i,j+1} = \ln r_{i-1,j} + \theta_i h + \sigma\sqrt{h}.$$

Note that $r_{i,j}$ can be expressed in terms of $r_{0,0}$:

$$\ln r_{i,j} = \ln r_{0,0} + h\sum_{k=1}^{i}\theta_k + [i - 2(i-j)]\sigma\sqrt{h}, \quad (16)$$

Note that in *Eq. (16)*, for j=0, $\ln r_{i,0} = \ln r_{0,0} + h\sum_{k=1}^{i}\theta_k - i\sigma\sqrt{h}$. Therefore, we can use as a baseline the interest rate in the ith period that is in the state j=0. We consider the parameter θ_i as crisp and that volatility is the unique argument determining $r_{i,j}$. Therefore, we can write $r_{i,j}$ as a function of σ, $r_{i,j}(\sigma)$, as follows:

$$\ln r_{i,j}(\sigma) = \ln r_{i,0}(\sigma) + 2j\sigma\sqrt{h} \longrightarrow r_{i,j}(\sigma) = r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}. \quad (17)$$

To value a contingent asset, the states of the short-term rates in every period i=1,2,…,n must be adjusted from the actual zero-coupon bond prices $P_1, P_2, \ldots, P_n$ and the annualized volatility σ. The binomial tree was fitted recursively using forward induction [70]. We calculate the price of an ADA in the ith period at the j^{th} state, which depends on the volatility; thus, we denote $d_{i,j}(\sigma)$ by

$$d_{ij}(\sigma) = \frac{1}{2}\left(d_{i-1,j-1}(\sigma)e^{-h\cdot r_{i-1,0}(\sigma)\cdot e^{2(j-1)\sigma\sqrt{h}}} + d_{i-1,j}(\sigma)e^{-h\cdot r_{i-1,0}(\sigma)\cdot e^{2j\sigma\sqrt{h}}}\right). \quad (18)$$

Then, from *Eq. (14)* and *Eq. (17)*, $r_{i,0}(\sigma)$ must be obtained from the equation equaling the actual value of of the zero-coupon bond and its theoretical value in the binomial lattice:

$$P_i - \sum_{j=0}^{i-1} d_{i,j}(\sigma) \cdot e^{-h\cdot r_{i,0}(\sigma)\cdot e^{2j\sigma\sqrt{h}}} = 0. \quad (19)$$

Unlike other interest rate movement models such as [69], where the interest rate is explicitly defined as a function of volatility $r_{i,0}(\sigma)$ through a closed formula, in the case of BDT, $r_{i,0}(\sigma)$ is implicitly defined so that for specific values of σ, it must be solved using algorithms such as Newton–Raphson [71].

Numerical application 2

Using the data used in numerical application 1, we induce the evolution of one-year interest rates (h=1) for i=0,1...,5 on May 30, 2023, in the European market for bond-rated AAA. This rating represents the highest possible credit quality and is commonly perceived by the market as virtually free of credit risk. Therefore, we assume a risk-free market environment for the purposes of applying the BDT model.. It should be noted that in numerical application 1, we verified that $r_{0.0}$=3.212% and σ=0.714%. Additionally, on that date, the spot rates, prices of zero coupon bonds, and one-year forward rates are recorded, as indicated in *Table 1*. In this regard, note that in *Table 1*, the prices of zero-coupon bonds are $P_i = e^{-R_i \cdot i}$, and the one-year forward yield rates are $r_i = R_{i+1} \cdot (i+1) - R_i \cdot i$.

Table 1. Prices in the European Union of AAA zero-coupon bonds on 30 May 2023.

Matutity (Years)	Spot Rate (R_i)	Price (P_i)	One Year- Forward Rates (r_i)
1	3.212%	0.9684	3.108%
2	3.160%	0.9388	3.021%
3	3.114%	0.9108	2.581%
4	2.981%	0.8876	2.649%
5	2.914%	0.8644	2.807%
6	2.896%	0.8405	

Table 2. Binomial lattices of the one-year rates and the prices of arrow-debrei assets in the European Union market of AAA zero-coupon bonds with the BDT model (30 May 2023).

	Binomial Lattices of One-Year Rates							Binomial Lattice of ADA Prices					
j|i	0	1	2	3	4	5		0	1	2	3	4	5
0	3.212%	3.086%	2.978%	2.516%	2.516%	2.692%	0	1.000	0.48419	0.23464	0.11378	0.05541	0.02697
1		3.130%	3.021%	2.552%	2.552%	2.731%	1		0.48419	0.46938	0.34143	0.22173	0.13487
2			3.065%	2.589%	2.635%	2.770%	2			0.23474	0.34158	0.33278	0.26994
3				2.626%	2.673%	2.810%	3				0.11393	0.22202	0.27018
4					2.711%	2.851%	4					0.05555	0.13522
5						2.892%	5						0.02707

3.2|Pricing a Call and Put European Options on One-Year Period Zero Coupon Bonds with the Black-Derman-Toy Model

Let be a contingent asset maturing in the ith period that will pay cash flow $c_{i,j}$ in state j, which, being an asset sensitive to interest rates, is a function of $r_{i,j}(\sigma)$. We consider, as in original BDT model, that cash-flows are not sensitive to other risks (e.g. credit or operational risks). We denote the function that determines cash flows as $c_{i,j} = c\left(r_{i,j}(\sigma)\right)$. Therefore, the price of such an asset is a function $\Pi_i(\sigma)$ and from *Eq. (13)*,

$$\Pi_i(\sigma) = \sum_{j=0}^{i} c\left(r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}\right) \cdot d_{ij}(\sigma). \quad (20)$$

We now price call and put European options with maturity $i \cdot h$ and strike rate Ki on a one-period zero coupon bond that pays 1 in $(i+1)\cdot h$. Let us calculate the price of that bond in the ith period in the j^{th} state as

$$\rho_{ij}(\sigma) = e^{-r_{i,j}(\sigma)\cdot h} = e^{-\left(r_{i,0}(\sigma)+2j\sigma\sqrt{h}\right)\cdot h}. \quad (21)$$

Then, the benefit obtained by a buyer of a call option on the one-year zero coupon bond in the ith period and jth state is $\max\{\rho_{ij}(\sigma) - K_i, 0\}$, which implies a cash flow:

$$c\left(r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}\right) = \max\left\{e^{-\left(r_{i,0}(\sigma)+2j\sigma\sqrt{h}\right)h} - K_i, 0\right\}, \quad (22)$$

Thus, the price of a call option is given by *Eqs.(13)* and *(22)*.

$$\Pi_i^c(\sigma) = \sum_{j=0}^{i} \max\left\{e^{-\left(r_{i,0}(\sigma)+2j\sigma\sqrt{h}\right)h} - K_i, 0\right\} \cdot d_{ij}(\sigma). \quad (23)$$

Analogously, the benefit obtained by a buyer of a put option is $\max\{K_i - \rho_{ij}(\sigma), 0\}$, whose corresponding cash flow from *Eq. (21)* is

$$c_{i,j}^p\left(r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}\right) = \max\left\{K_i - e^{-\left(r_{i,0}(\sigma)+2j\sigma\sqrt{h}\right)h}, 0\right\}, \quad (24)$$

and then, from *Eq. (13)* and *Eq. (24)*:

$$\Pi_i^p(\sigma) = \sum_{j=0}^{i} \max\left\{K_i - e^{-\left(r_{i,0}(\sigma)+2j\sigma\sqrt{h}\right)h}, 0\right\} \cdot d_{ij}(\sigma). \quad (25)$$

Numerical application 3

In the same market and date as in the above numerical applications, we obtain the prices of European call and put options on one-year zero-coupon bonds exercised at i=1,2...,4. The strike prices are set based on the one-year forward rate given by $K_i = e^{-r_i}$, as indicated in *Table 1*. Cash flows are calculated using *Eq. (22)* for call options and *Eq. (24)* for puts. The prices are determined using *Eqs. (23)* and *(25)*. The results in *Tables 3* and *4* are based on a notional value of 100,000.

Table 3. Cash-flows and prices of call options on one-year zero coupon bonds.

Ki=	96939.93	97023.89	97451.56	97385.93
i (Year of Exercise)	1	2	3	4
j=0	21.56	41.81	63.95	86.16
j=1	0.00	0.24	28.67	50.27
j=2		0.00	0.00	13.88
j=3			0.00	0.00
j=4				0.00
$\boldsymbol{\Pi_i^c(0.714\%)}$	10.44	9.92	17.06	20.54

Note: Prices are calculated over 100,000 Mus.

Table 4. Cash-flows and prices of put options on one-year zero coupon bonds.

Ki=	96939.93	97023.89	97451.56	97385.93
i (year of exercise)	1	2	3	4
j=0	0.00	0.00	0.00	0.00
j=1	21.46	0.00	0.00	0.00
j=2		41.91	7.10	0.00
j=3			43.38	23.02
j=4				60.43
$\mathbf{\Pi_i^p(0.714\%)}$	10.39	9.84	7.37	8.47

Note: Prices are calculated over 100,000 Mus.

4 | Pricing Options on Bonds with an Intuitionistic Fuzzy Black-Derman-Toy Model of a Temporal Structure of Interest Rates

4.1 | Fitting an Intuitionistic Fuzzy Extension of the Black-Derman-Toy Model

We now assume that the volatility of short-term yield rates *Eqs. (15)* and *(16)* are determined by IFN $\tilde{\sigma}^I$, which was obtained with *Eqs. (9a)–(9d)* in *Remark 12*. Thus, the $\langle\alpha,\beta\rangle$-cuts are $\sigma_{\langle\alpha,\beta\rangle} = \langle\sigma_\alpha = \left[\underline{\sigma_\alpha}, \overline{\sigma_\alpha}\right], \sigma'_\beta = \left[\underline{\sigma'_\beta}, \overline{\sigma'_\beta}\right], 0\leq\alpha+\beta\leq1, \alpha,\beta\in(0,1)\rangle$.

The hypothesis of fuzzy volatility quantified with a type-1 FNs is significantly extended in FROP in discrete time, and this also applies in a fixed income market setting. It is usually assumed that only volatility is fuzzy [27], [30], [31], [38], [39], [72]. In this thesis, the binomial lattice representation in *Fig. 4* becomes an IFN binomial tree, whose outcomes are as follows:

$$\tilde{r}^I_{i,j} = \tilde{r}^I_{i,0} \cdot e^{2j\tilde{\sigma}^I\sqrt{h}}.$$

We subsequently must fit $\tilde{r}^I_{i,j}$ by means of $r_{i,j_{\langle\alpha,\beta\rangle}} = \langle r_{i,j_\alpha} = \left[\underline{r_{i,j_\alpha}}, \overline{r_{i,j_\alpha}}\right], r_{i,j'_\beta} = \left[\underline{r_{i,j'_\beta}}, \overline{r_{i,j'_\beta}}\right], 0\leq\alpha+\beta\leq1, \alpha,\beta\in(0,1)\rangle$, j=0,1,…,i. This process is performed in three steps: First, $r_{i,0_{\langle\alpha,\beta\rangle}}$ is defined; second, $r_{i,j_{\langle\alpha,\beta\rangle}}$ is fit, j=1,…,i-1; and finally, $r_{i,i_{\langle\alpha,\beta\rangle}}$ are adjusted. Therefore, we describe these steps in detail below.

Step 1. To fit $r_{i,0_{\langle\alpha,\beta\rangle}}$, we must consider that the highest volatilities generate greater variability in *Fig. 4*, and thus, more sparcity in the three. Therefore, for $r_{i,0}$, it must be accomplished that $\frac{\partial r_{i,0}}{\partial\sigma} < 0$, and for $r_{i,i}$, $\frac{\partial r_{i,i}}{\partial\sigma} > 0$. Therefore, by using *Eqs. (11d)–(11 g)*, we can state that:

$$r_{i,0_\alpha} = \left[\underline{r_{i,0_\alpha}}, \overline{r_{i,0_\alpha}}\right] = \left[r_{i,0}(\overline{\sigma_\alpha}), r_{i,0}\left(\underline{\sigma_\alpha}\right)\right]. \quad \textbf{(27a)}$$

$$r_{i,0'_\beta} = \left[\underline{r_{i,0'_\beta}}, \overline{r_{i,0'_\beta}}\right] = \left[r_{i,0}\left(\overline{\sigma'_\beta}\right), r_{i,0}\left(\underline{\sigma'_\beta}\right)\right]. \quad \textbf{(27b)}$$

Note that the extremes of $r_{i,0_\alpha}$ and $r_{i,0'_\beta}$ must be obtained from the implicit *Function (19)*, which is an equation similar to that needed to obtain the Internal Rate of Return (IRR) with one change of sign in the cash-flows. By analogy to that proposed in [73] to solve the IRR equation with fuzzy parameters to fit these extremes, we have to solve.

$$P_i - \sum_{j=0}^{i-1} d_{i,j}(\overline{\sigma_\alpha}) \cdot e^{-h\cdot\underline{r_{i,0_\alpha}}\cdot e^{2j\overline{\sigma_\alpha}\sqrt{h}}} = 0 \text{ and } P_i - \sum_{j=0}^{i-1} d_{i,j}\left(\underline{\sigma_\alpha}\right) \cdot e^{-h\cdot\overline{r_{i,0_\alpha}}\cdot e^{2j\underline{\sigma_\alpha}\sqrt{h}}} = 0. \quad \textbf{(28a)}$$

$$P_i - \sum_{j=0}^{i-1} d_{i,j}\left(\overline{\sigma'_\beta}\right) \cdot e^{-h\cdot\underline{r_{i,0'_\beta}}\cdot e^{2j\overline{\sigma'_\beta}\sqrt{h}}} = 0 \text{ and } P_i - \sum_{j=0}^{i-1} d_{i,j}\left(\underline{\sigma'_\beta}\right) \cdot e^{-h\cdot\overline{r_{i,0'_\beta}}\cdot e^{2j\underline{\sigma'_\beta}\sqrt{h}}} = 0. \quad \textbf{(28b)}$$

Step 2. For j=1,2,…,i-1, $r_{i,j_{\langle\alpha,\beta\rangle}}$ must be fitted using *Eqs. (11b)–(11c)*. In the case of r_{i,j_α}, for the lower extreme, $\underline{r_{i,j_\alpha}} = r_{i,j}(\sigma^-) = r_{i,0}(\sigma^-) \cdot e^{2j\sigma^-\sqrt{h}}$ is:

$$\sigma^- = \underset{\sigma}{\operatorname{argmin}}\, r_{i,j}(\sigma) = r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}, s.t. \sigma \in \sigma_\alpha. \quad \textbf{(29a)}$$

and analogously, the upper extreme of r_{i,j_α} is obtained as $\overline{r_{i,j_\alpha}} = r_{i,j}(\sigma^+) = r_{i,0}(\sigma^+) \cdot e^{2j\sigma^+\sqrt{h}}$, where:

$$\sigma^+ = \underset{\sigma}{\operatorname{argmax}}\, r_{i,j}(\sigma) = r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}, s.t. \sigma \in \sigma_\alpha. \quad \textbf{(29b)}$$

Then, we proceed analogously to state $r_{i,j}{}'_\beta$. Then, for the lower extreme, we find $\underline{r_{i,j}{}'_\beta} = r_{i,j}(\sigma'^-) = r_{i,0}(\sigma'^-) \cdot e^{2j\sigma'^-\sqrt{h}}$ to be:

$$\sigma'^- = \underset{\sigma}{\operatorname{argmin}}\, r_{i,j}(\sigma) = r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}, s.t. \sigma \in \sigma'_\beta. \quad \textbf{(29c)}$$

And subsequently, the upper extreme of $r_{i,j}{}'_\beta$ is obtained as $\overline{r_{i,j}{}'_\beta} = r_{i,j}(\sigma'^+) = r_{i,0}(\sigma'^+) \cdot e^{2j\sigma'^+\sqrt{h}}$, where:

$$\sigma'^+ = \underset{\sigma}{\operatorname{argmax}}\, r_{i,j}(\sigma) = r_{i,0}(\sigma) \cdot e^{2j\sigma\sqrt{h}}, s.t. \sigma \in \sigma'_\beta. \quad \textbf{(29d)}$$

Step 3. In the case of j=i, given that $\frac{\partial r_{i,i}}{\partial \sigma} > 0$, $r_{i,i_{\langle\alpha,\beta\rangle}}$ is delimited by

$$r_{i,i_\alpha} = \left[\underline{r_{i,i_\alpha}}, \overline{r_{i,i_\alpha}}\right] = \left[r_{i,i}\left(\underline{\sigma_\alpha}\right), r_{i,i}(\overline{\sigma_\alpha})\right] = \left[r_{i,0}\left(\underline{\sigma_\alpha}\right) \cdot e^{2i\underline{\sigma_\alpha}\sqrt{h}}, r_{i,0}(\overline{\sigma_\alpha}) \cdot e^{2i\overline{\sigma_\alpha}\sqrt{h}}\right]. \quad \textbf{(30a)}$$

and,

$$r_{i,i}{}'_\beta = \left[\underline{r_{i,i}{}'_\beta}, \overline{r_{i,i}{}'_\beta}\right] = \left[r_{i,i}\left(\underline{\sigma'_\beta}\right), r_{i,i}\left(\overline{\sigma'_\beta}\right)\right] = \left[r_{i,0}\left(\underline{\sigma'_\beta}\right) \cdot e^{2i\underline{\sigma'_\beta}\sqrt{h}}, r_{i,0}\left(\overline{\sigma'_\beta}\right) \cdot e^{2i\overline{\sigma'_\beta}\sqrt{h}}\right]. \quad \textbf{(30b)}$$

Although volatility is modeled as an IFN, the Newton–Raphson method is applied separately to the endpoints of each α- and β-cut of the volatility interval to fit $r_{i,0_\alpha}$, $r_{i,0}{}'_\beta$, r_{i,i_α} and $r_{i,i}{}'_\beta$. This avoids the need for generalized fuzzy arithmetic in root-finding and ensures convergence under classical conditions (i.e., continuity and differentiability of the function).

It is worth noting that the structure of the resulting equation is analogous to that of IRR estimation, with a single sign change in the cash flows. Therefore, the application of solution techniques commonly adopted in fuzzy IRR estimation becomes straightforward and immediate (see [73]).

Likewise, in r_{i,j_α} and $r_{i,j}{}'_\beta$, j=1,2,…,i-1 each endpoint of can be computed by solving a crisp optimization problem, where the objective function depends on the discretized possible values of the fuzzy volatility. Specifically, for each α-level (or β-level), the minimum and maximum of the short-term interest rate function are determined over the discretized support of the corresponding fuzzy volatility in order to identify the bounds of the interest rate in period i, scenario j.

This procedure is equivalent to solving multiple instances of classical yield-matching problems, structurally analogous to IRR calculations, ciated cash flow structure, each IRR instance admits a unique solution, which further simplifies computation.

Based on [72] for computing fuzzy functions, we can assert that if the interest rate is monotonic with respect to volatility within scenario j, then the extrema in *Eqs. (29c)* and *(29d)* are attained at the endpoints of σ_α and σ'_β.

However, if monotonicity does not hold, the extrema of the resulting intervals r_{i,j_α} and $r_{i,j}{}'_\beta$, j=1,2,…,i-1 may lie at interior points of σ_α and σ'_β, respectively. In such cases, a discretized search is performed over the support of the fuzzy volatility.

The resolution precision depends on the granularity of this discretization: the finer the grid, the more accurate the resulting bounds, albeit at the cost of increased computational complexity. Nonetheless, this approach remains computationally tractable and avoids the complications of symbolic fuzzy arithmetic, while staying consistent with the semantics of intuitionistic fuzzy modeling.

Numerical application 4

We now estimate the term structure of intuitionistic interest rates derived from the estimated volatility through IFNs in numerical application 1, and the prices of zero-coupon bonds in *Table 1*. Thus, while the confidence intervals of $r_{i,j_{\langle\alpha,\beta\rangle}}$ derived from the membership function, $r_{i,j_{\alpha}}$, are provided in *Tables 5* and *6* shows those from the non-membership function $r_{i,j}{}'_{\beta}$.

Therefore, for i=5, $\tilde{r}^{I}_{5,j}$ j=0,1,2,…,5, by mixing *Table 5* and *Table 6*, we can check that for the one-year interest in the state j=0, $r_{5,0_{\langle 1,0\rangle}} = \langle r_{5,0_1} = [2.692\%, 2.692\%], r_{5,0}{}'_0 = [2.692\%, 2.692\%]\rangle$, and $r_{5,0_{\langle 0,1\rangle}} = \langle r_{5,0_0} = [2.681\%, 2.701\%], r_{5,0}{}'_1 = [2.675\%, 2.705\%]\rangle$.

Thus, the information available to the analyst regarding the one-year interest rate being smaller at i=5 is certainly more comprehensive than that of simple point estimation ($r_{5,0}$ =2.692%) or through a confidence interval for a specific level of significance, such as the one associated with a significance level of 5%, which would be [2.681% and 2.701%]. The most plausible interest rates in this period (i=5) and scenario (j=0) had a maximum reliability interest rate of 2.692%.

Furthermore, while the membership function would indicate possible values ranging between 2.681% and 2.701%, it is also explicitly stated that, in no case is the one-year interest rate less than 2.675% or greater than 2.705%.

Similarly, for i=5 and j=5, it can be checked that $r_{5,5_{\langle 1,0\rangle}} = \langle r_{5,5_1} = [2.892\%, 2.892\%], r_{5,5}{}'_0 = [2.892\%, 2.892\%]\rangle$, and $r_{5,5_{\langle 0,1\rangle}} = \langle r_{5,5_0} = [2.885\%, 2.9\%], r_{5,5}{}'_1 = [2.881\%, 2.904\%]\rangle$, that is, while there is a reasonable expectation that the interest rate $r_{5,5}$ may vary between [2.885% and 2.9%], it is also expected that this yield rate must not be under 2.881% or above 2.904%.

Table 5. The values of the α-cuts of $\tilde{r}^{I}_{i,j}$, r_{i,j_α} of one-year interest rates in the European fixed income market of bonds with AAA rating (May 30, 2023).

i		0		1		2		3		4		5	
j	α	$\underline{r_{0,0_\alpha}}$	$\overline{r_{0,0_\alpha}}$	$\underline{r_{1,0_\alpha}}$	$\overline{r_{1,0_\alpha}}$	$\underline{r_{2,0_\alpha}}$	$\overline{r_{2,0_\alpha}}$	$\underline{r_{3,0_\alpha}}$	$\overline{r_{3,0_\alpha}}$	$\underline{r_{4,0_\alpha}}$	$\overline{r_{4,0_\alpha}}$	$\underline{r_{5,0_\alpha}}$	$\overline{r_{5,0_\alpha}}$
	1	3.212	3.212	3.086	3.086	2.978	2.978	2.516	2.516	2.561	2.561	2.692	2.692
	0.75	3.212	3.212	3.085	3.086	2.977	2.979	2.514	2.517	2.558	2.562	2.689	2.694
0	0.5	3.212	3.212	3.085	3.087	2.976	2.980	2.513	2.519	2.556	2.564	2.687	2.697
	0.25	3.212	3.212	3.084	3.087	2.975	2.981	2.511	2.520	2.554	2.566	2.684	2.699
	0	3.212	3.212	3.084	3.087	2.974	2.982	2.510	2.521	2.552	2.568	2.681	2.701
				$\underline{r_{1,0_\alpha}}$	$\overline{r_{1,0_\alpha}}$	$\underline{r_{2,1_\alpha}}$	$\overline{r_{2,1_\alpha}}$	$\underline{r_{3,1_\alpha}}$	$\overline{r_{3,1_\alpha}}$	$\underline{r_{4,1_\alpha}}$	$\overline{r_{4,1_\alpha}}$	$\underline{r_{5,1_\alpha}}$	$\overline{r_{5,1_\alpha}}$
	1			3.130	3.130	3.021	3.021	2.552	2.552	2.597	2.597	2.731	2.731
	0.75			3.130	3.131	3.021	3.021	2.551	2.553	2.596	2.598	2.729	2.732
1	0.5			3.129	3.131	3.021	3.021	2.551	2.553	2.595	2.599	2.727	2.734
	0.25			3.129	3.132	3.021	3.021	2.550	2.554	2.594	2.601	2.726	2.735
	0			3.128	3.132	3.021	3.021	2.549	2.555	2.593	2.602	2.724	2.737
						$\underline{r_{2,2_\alpha}}$	$\overline{r_{2,2_\alpha}}$	$\underline{r_{3,2_\alpha}}$	$\overline{r_{3,2_\alpha}}$	$\underline{r_{4,2_\alpha}}$	$\overline{r_{4,2_\alpha}}$	$\underline{r_{5,2_\alpha}}$	$\overline{r_{5,2_\alpha}}$
	1					3.065	3.065	2.589	2.589	2.635	2.635	2.770	2.770
	0.75					3.064	3.066	2.589	2.589	2.634	2.635	2.769	2.771
2	0.5					3.063	3.067	2.589	2.589	2.634	2.635	2.768	2.772
	0.25					3.062	3.068	2.588	2.589	2.634	2.636	2.768	2.772
	0					3.061	3.069	2.588	2.590	2.633	2.636	2.767	2.773
								$\underline{r_{3,3_\alpha}}$	$\overline{r_{3,3_\alpha}}$	$\underline{r_{4,3_\alpha}}$	$\overline{r_{4,3_\alpha}}$	$\underline{r_{5,3_\alpha}}$	$\overline{r_{5,3_\alpha}}$
	1							2.626	2.626	2.673	2.673	2.810	2.810
	0.75							2.625	2.627	2.672	2.673	2.810	2.810
3	0.5							2.624	2.628	2.672	2.674	2.810	2.810
	0.25							2.623	2.629	2.671	2.674	2.810	2.810
	0							2.623	2.630	2.671	2.675	2.810	2.810
										$\underline{r_{4,4_\alpha}}$	$\overline{r_{4,4_\alpha}}$	$\underline{r_{5,4_\alpha}}$	$\overline{r_{5,4_\alpha}}$
	1									2.711	2.711	2.851	2.851
	0.75									2.710	2.713	2.850	2.852
4	0.5									2.709	2.714	2.849	2.853
	0.25									2.707	2.716	2.848	2.854
	0									2.706	2.717	2.847	2.855
												$\underline{r_{5,5_\alpha}}$	$\overline{r_{5,5_\alpha}}$
	1											2.892	2.892
	0.75											2.890	2.894
5	0.5											2.888	2.896
	0.25											2.886	2.898
	0											2.885	2.900

Note: interest rates are expressed as percentages.

Table 6. The values of the β-cuts of $\tilde{r}^{I}_{i,j}$ of one-year interest rates in the European Union bond market of bonds with AAA ratings (May 30, 2023).

i		0		1		2		3		4		5	
j	β	$\underline{r_{0,0}}'_{\beta}$	$\overline{r_{0,0}}'_{\beta}$	$\underline{r_{1,0}}'_{\beta}$	$\overline{r_{1,0}}'_{\beta}$	$\underline{r_{2,0}}'_{\beta}$	$\overline{r_{2,0}}'_{\beta}$	$\underline{r_{3,0}}'_{\beta}$	$\overline{r_{3,0}}'_{\beta}$	$\underline{r_{4,0}}'_{\beta}$	$\overline{r_{4,0}}'_{\beta}$	$\underline{r_{5,0}}'_{\beta}$	$\overline{r_{5,0}}'_{\beta}$
	0	3.212	3.212	3.086	3.086	2.978	2.978	2.516	2.516	2.561	2.561	2.692	2.692
	0.25	3.212	3.212	3.085	3.086	2.977	2.980	2.514	2.518	2.557	2.563	2.688	2.695
0	0.5	3.212	3.212	3.084	3.087	2.975	2.981	2.511	2.520	2.554	2.566	2.684	2.699
	0.75	3.212	3.212	3.083	3.088	2.973	2.982	2.509	2.522	2.551	2.568	2.680	2.702
	1	3.212	3.212	3.082	3.088	2.972	2.983	2.506	2.524	2.548	2.571	2.675	2.705
	β			$\underline{r_{1,1}}'_{\beta}$	$\overline{r_{1,1}}'_{\beta}$	$\underline{r_{2,1}}'_{\beta}$	$\overline{r_{2,1}}'_{\beta}$	$\underline{r_{3,1}}'_{\beta}$	$\overline{r_{3,1}}'_{\beta}$	$\underline{r_{4,1}}'_{\beta}$	$\overline{r_{4,1}}'_{\beta}$	$\underline{r_{5,1}}'_{\beta}$	$\overline{r_{5,1}}'_{\beta}$
	0			3.130	3.130	3.021	3.021	2.552	2.552	2.597	2.597	2.731	2.731
	0.25			3.129	3.131	3.021	3.021	2.551	2.553	2.595	2.599	2.728	2.733
1	0.5			3.129	3.132	3.021	3.021	2.550	2.554	2.594	2.600	2.725	2.735
	0.75			3.128	3.133	3.021	3.021	2.549	2.555	2.592	2.602	2.722	2.738
	1			3.127	3.133	3.021	3.021	2.548	2.556	2.590	2.603	2.720	2.740
	β					$\underline{r_{2,2}}'_{\beta}$	$\overline{r_{2,2}}'_{\beta}$	$\underline{r_{3,2}}'_{\beta}$	$\overline{r_{3,2}}'_{\beta}$	$\underline{r_{4,2}}'_{\beta}$	$\overline{r_{4,2}}'_{\beta}$	$\underline{r_{5,2}}'_{\beta}$	$\overline{r_{5,2}}'_{\beta}$
	0					3.065	3.065	2.589	2.589	2.635	2.635	2.770	2.770
	0.25					3.063	3.066	2.589	2.589	2.634	2.635	2.769	2.771
2	0.5					3.062	3.068	2.589	2.589	2.634	2.636	2.767	2.772
	0.75					3.061	3.070	2.588	2.590	2.633	2.636	2.766	2.773
	1					3.059	3.071	2.588	2.590	2.633	2.636	2.765	2.775
	β							$\underline{r_{3,3}}'_{\beta}$	$\overline{r_{3,3}}'_{\beta}$	$\underline{r_{4,3}}'_{\beta}$	$\overline{r_{4,3}}'_{\beta}$	$\underline{r_{5,3}}'_{\beta}$	$\overline{r_{5,3}}'_{\beta}$
	0							2.626	2.626	2.673	2.673	2.810	2.810
	0.25							2.625	2.628	2.672	2.674	2.810	2.810
3	0.5							2.624	2.630	2.671	2.674	2.810	2.810
	0.75							2.622	2.631	2.671	2.675	2.810	2.810
	1							2.621	2.633	2.670	2.676	2.810	2.810
	β									$\underline{r_{4,4}}'_{\beta}$	$\overline{r_{4,4}}'_{\beta}$	$\underline{r_{5,4}}'_{\beta}$	$\overline{r_{5,4}}'_{\beta}$
	0									2.711	2.711	2.851	2.851
	0.25									2.709	2.714	2.849	2.852
4	0.5									2.707	2.716	2.848	2.854
	0.75									2.706	2.718	2.847	2.855
	1									2.704	2.721	2.845	2.857
	β											$\underline{r_{5,5}}'_{\beta}$	$\overline{r_{5,5}}'_{\beta}$
	0											2.892	2.892
	0.25											2.889	2.895
5	0.5											2.886	2.898
	0.75											2.884	2.901
	1											2.881	2.904

Note: interest rates are expressed as percentages.

While the intuitionistic extension of the BDT model increases the number of computations due to the evaluation of ⟨α, β⟩-cuts and interval-valued rates, the model remains computationally tractable for realistic time horizons and discretization levels. For example, in the numerical application presented (5 periods and 5 <α,β>-levels), the full fuzzy lattice, constructed using iterative IRR estimation at each node, is completed in under three seconds using standard R functions such as uniroot, with the fuzzy volatility discretized in increments of 0.02%. This implementation was run on a personal computer equipped with an 11th Gen Intel Core i5-1135G7 processor (8 logical cores, 2.40 GHz base frequency). The efficiency stems from the smooth structure of the underlying functions, each involving a cash flow sequence with a single sign change, ensuring rapid convergence. Additionally, since each node and cut-level computation is independent, parallelization (e.g., via the parallel or future packages) can reduce total runtime to around one second. Although the theoretical complexity is exponential, the recombination of nodes and bounded depth of the binomial tree ensure that execution times remain manageable, making the approach suitable for practical financial modeling.

4.2|Pricing Options on Bonds with an Intuitionistic Fuzzy Black-Derman-Toy Binomial Lattice

We now price a contingent asset that is sensible to interest rates using the analytical framework described in Sections 3 and 4.1. The price of an asset that pays a cash flow sensible to interest rates in the ith period is a function of IFN volatility $\tilde{\sigma}^{I}$ whose price is obtained by evaluating *Eq. (20)*:

$$\tilde{\Pi}_i^I = \Pi_i(\tilde{\sigma}^I) = \sum_{j=0}^{i} c\left(r_{i,0}(\tilde{\sigma}^I) \cdot e^{2j\tilde{\sigma}^I\sqrt{h}}\right) \cdot d_{ij}\left(\tilde{\sigma}^I\right). \tag{31}$$

Therefore, $\tilde{\Pi}_i^I$ is obtained throughout level sets $\Pi_{i\langle\alpha,\beta\rangle} = \langle \Pi_{i_\alpha} = \left[\underline{\Pi_{i_\alpha}}, \overline{\Pi_{i_\alpha}}\right], \Pi'_{i\beta} = \left[\underline{\Pi'_{i\beta}}, \overline{\Pi'_{i\beta}}\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in (0,1) \rangle$:

$$\Pi_{i\langle\alpha,\beta\rangle} = \Pi_i(\sigma_{\langle\alpha,\beta\rangle}) = \sum_{j=0}^{i} c\left(r_{i,0}(\sigma_{\langle\alpha,\beta\rangle}) \cdot e^{2j\sigma_{\langle\alpha,\beta\rangle}\sqrt{h}}\right) \cdot d_{ij}\left(\sigma_{\langle\alpha,\beta\rangle}\right), \tag{32}$$

Which can be evaluated with *Eqs. (11d)-(11g)*.

In the case of European options on one-period zero coupon bonds that must be exercised in the ith period, $\Pi_{i\langle\alpha,\beta\rangle}$ can be obtained by considering that the option value is positively linked to volatility [4]. These are contracts in which both the strike price and the maturity date are explicitly defined in the contractual terms. Therefore, it would not be meaningful to introduce uncertainty into these parameters. Volatility remains the sole source of uncertainty in the option pricing process. This is a common assumption in fuzzy binomial option pricing models applied to stocks [20], [28] and interest rate derivatives [38], [39].Thus, from *Eqs. (11d)-(11g)*, and *Eq. (20)*,

$$\Pi_{i_\alpha} = \left[\underline{\Pi_{i_\alpha}}, \overline{\Pi_{i_\alpha}}\right] = \left[\sum_{j=0}^{i} c\left(r_{i,0}\left(\underline{\sigma_\alpha}\right) \cdot e^{2j\underline{\sigma_\alpha}\sqrt{h}}\right) \cdot d_{ij}\left(\underline{\sigma_\alpha}\right), \sum_{j=0}^{i} c\left(r_{i,0}(\overline{\sigma_\alpha}) \cdot e^{2i\overline{\sigma_\alpha}\sqrt{h}}\right) \cdot d_{ij}\left(\overline{\sigma_\alpha}\right)\right]. \tag{33a}$$

$$\Pi_i{}'_\beta = \left[\underline{\Pi_i{}'_\beta}, \overline{\Pi_i{}'_\beta}\right] = \left[\sum_{j=0}^{i} c\left(r_{i,0}\left(\underline{\sigma'_\beta}\right) \cdot e^{2j\underline{\sigma'_\beta}\sqrt{h}}\right) \cdot d_{ij}\left(\underline{\sigma'_\beta}\right), \sum_{j=0}^{i} c\left(r_{i,0}\left(\overline{\sigma'_\beta}\right) \cdot e^{2i\overline{\sigma'_\beta}\sqrt{h}}\right) \cdot d_{ij}\left(\overline{\sigma'_\beta}\right)\right]. \tag{33b}$$

Thus, for a European call option exercitable in the ith period for a bond maturing in the (i+1)th period, the price $\tilde{\Pi}_i^{I,c}$ can be characterized through $\Pi^c_{i\langle\alpha,\beta\rangle}$ with *Eqs. (23)*, *(33a)*, and *(33b)* as

$$\Pi^c_{i_\alpha} = \left[\underline{\Pi^c_{i_\alpha}}, \overline{\Pi^c_{i_\alpha}}\right] = \left[\Pi^c_i\left(\underline{\sigma_\alpha}\right), \Pi^c_i(\overline{\sigma_\alpha})\right] = \\ = \left[\sum_{j=0}^{i} \max\left\{e^{-\left(r_{i,0}\left(\underline{\sigma_\alpha}\right)+2j\underline{\sigma_\alpha}\sqrt{h}\right)h} - K_i, 0\right\} \cdot d_{ij}\left(\underline{\sigma_\alpha}\right), \right. \tag{34a}$$

and,

$$\Pi^{c\prime}_{i_\beta} = \left[\underline{\Pi^{c\prime}_{i_\beta}}, \overline{\Pi^{c\prime}_{i_\beta}}\right] = \left[\Pi^c_i\left(\underline{\sigma'_\beta}\right), \Pi^c_i\left(\overline{\sigma'_\beta}\right)\right] = \\ = \left[\sum_{j=0}^{i} \max\left\{e^{-\left(r_{i,0}\left(\underline{\sigma'_\beta}\right)+2j\underline{\sigma'_\beta}\sqrt{h}\right)h} - K_i, 0\right\} \cdot d_{ij}\left(\underline{\sigma'_\beta}\right). \right. \tag{34b}$$

Similarly, for a European put option exercisable in the ith period for a bond with maturity in the (i+1)th period, price $\tilde{\Pi}_i^{I,p}$ is found analogously with *Eqs. (25)*, *(33a)*, and *(33b)* through $\Pi^p_{i\langle\alpha,\beta\rangle}$:

$$\Pi_{i_\alpha}^{p}=\left[\underline{\Pi_{i_\alpha}^{p}},\overline{\Pi_{i_\alpha}^{p}}\right]=\left[\Pi_{i}^{p}\left(\underline{\sigma_\alpha}\right),\Pi_{i}^{p}(\overline{\sigma_\alpha})\right]==\left[\sum_{j=0}^{i}\max\left\{K_i-e^{-\left(r_{i,0}\left(\underline{\sigma_\alpha}\right)+2j\underline{\sigma_\alpha}\sqrt{h}\right)h}-,0\right\}\cdot d_{ij}\left(\underline{\sigma_\alpha}\right),\sum_{j=0}^{i}\max\left\{K_i-e^{-\left(r_{i,0}(\overline{\sigma_\alpha})+2j\overline{\sigma_\alpha}\sqrt{h}\right)h},0\right\}\cdot d_{ij}\left(\overline{\sigma_\alpha}\right)\right]. \quad (35a)$$

and,

$$\Pi_{i_\beta}^{p\prime}=\left[\underline{\Pi_{i_\beta}^{p\prime}},\overline{\Pi_{i_\beta}^{p\prime}}\right]=\left[\Pi_{i}^{p}\left(\underline{\sigma'_\beta}\right),\Pi_{i}^{p}\left(\overline{\sigma'_\beta}\right)\right]=\left[\sum_{j=0}^{i}\max\left\{K_i-e^{-\left(r_{i,0}\left(\underline{\sigma'_\beta}\right)+2j\underline{\sigma'_\beta}\sqrt{h}\right)h},0\right\}\cdot d_{ij}\left(\underline{\sigma'_\beta}\right),\sum_{j=0}^{i}\max\left\{K_i-e^{-\left(r_{i,0}(\overline{\sigma'_\beta})+2j\overline{\sigma'_\beta}\sqrt{h}\right)h},0\right\}\cdot d_{ij}\left(\overline{\sigma'_\beta}\right)\right]. \quad (35b)$$

Numerical application 5

We now derive the IFN estimates of prices for European call and put options on one-year zero-coupon bonds to be exercised at i=1,2,...,4, using the IFN volatility adjusted in numerical application 1. In numerical applications, it has been used to fit the temporal structure of interest rates in numerical applications 4. The options priced in this numerical application are the same as those previously evaluated using crisp annualized volatility in numerical application 3 (see *Tables 3* and *4*). The strike prices are listed in *Tables 3* and *4*. The intuitionistic prices of the call options are determined using *Eqs. (34a)–(34b)* and are presented in *Table 7*, while those of the put options, in Table 8, are determined using *Eqs. (35a)–(35b)*. Once again, the values were based on a notional value of 100,000.

The use of IFNs in option pricing, apart from providing more comprehensive information than that associated with point estimates, intervals, or FNs, allows for intuitive interpretation by an analyst who is not familiar with fuzzy logic. Thus, $\langle\alpha,\beta\rangle$ −cut arithmetics allows for systematizing option price values based on the degree of possibility and nonpossibility of the evaluated prices.

For example, let us analyze the intuitionistic price of the European option $\widetilde{\Pi}_3^{I,c}$ in *Table 7*. The <1,0>-cut of the intuitionistic fuzzy volatility corresponds to a crisp value of $\sigma_{\langle1,0\rangle}$=0.714%, which, when used in the classical BDT model, yields an option price of 17.03 Mus. This is precisely the <1,0>-cut of the option price ($\Pi_{3<1,0>}^{c}$) in the intuitionistic fuzzy framework.

However, the <0,1>-cut of the volatility is given by $\sigma_{\langle0,1\rangle}=\langle\sigma_0=[0.657\%, 0.783\%], \sigma'_1=[0.630\%, 0.822\%]\rangle$, which reflects the full range of epistemic uncertainty. Through the application of intuitionistic fuzzy arithmetic (as defined in *Eqs. (26)-(35b)*, this range of volatility translates into an <0,1>-cut for the option price of $\Pi_{3<0,1>}^{c}=\langle[15.65, 18.63], [15.02, 19.55]\rangle$. Thus, equipping volatility with an intuitionistic fuzzy structure allows us to derive prices expressed as IFNs, capturing both plausible values and degrees of (non) membership under uncertainty.

The point estimate of the price was 17.03 MU, which can be considered the most reliable value. Considering the possibilistic uncertainty in volatility, which has been captured by adjusting it to a lower possibility distribution, whose support is the confidence interval for its sample estimate containing 95% of the possible volatility values, implies an estimate of the variability over the point value, leading to price fluctuations between 15.65 and 18.63. These values can be considered as two reasonably extreme price scenarios (lower and upper).

Finally, introducing intuitionistic uncertainty considering the 99.7 rule allows us to obtain in the nonmembership of $\widetilde{\Pi}_3^{I,c}$, $\Pi_{3_1}^{c\prime}$=[15.02, 19.55]. In other words, the prices associated with scenarios that can be

assimilated into the completely extreme scenario are 15.02 for the lower scenario and 19.55 for the upper scenario.

In sum, the proposed framework can be interpreted as a structured methodology for analyzing the sensitivity of derivative prices by explicitly specifying preferred levels of membership and non-membership. These functions are governed by the intuitionistic representation of uncertainty, which is anchored to the 95% and 99.7% percentiles (i.e., those that are used to fix γ and γ*), which are widely used in financial practice to represent reasonably extreme (95%) and highly extreme (99.7%) adverse scenarios [57], [58].

Table 7. Intuitionistic fuzzy prices of European call option prices on one-year zero coupon bonds.

		i=1				i=2				i=3				i=4			
α	β	$\underline{\Pi^{c}_{1_\alpha}}$	$\overline{\Pi^{c}_{1_\alpha}}$	$\underline{\Pi^{c'}_{1_\beta}}$	$\overline{\Pi^{c'}_{1_\beta}}$	$\underline{\Pi^{c}_{2_\alpha}}$	$\overline{\Pi^{c}_{2_\alpha}}$	$\underline{\Pi^{c'}_{2_\beta}}$	$\overline{\Pi^{c'}_{2_\beta}}$	$\underline{\Pi^{c}_{3_\alpha}}$	$\overline{\Pi^{c}_{3_\alpha}}$	$\underline{\Pi^{c'}_{3_\beta}}$	$\overline{\Pi^{c'}_{3_\beta}}$	$\underline{\Pi^{c}_{4_\alpha}}$	$\overline{\Pi^{c}_{4_\alpha}}$	$\underline{\Pi^{c'}_{4_\beta}}$	$\overline{\Pi^{c'}_{4_\beta}}$
1	0	10.43	10.43	10.43	10.43	9.87	9.87	9.87	9.87	17.03	17.03	17.03	17.03	20.48	20.48	20.48	20.48
0.75	0.25	10.29	10.57	10.28	10.58	9.74	10.00	9.73	10.01	16.81	17.26	16.80	17.28	20.20	20.75	20.19	20.77
0.5	0.5	10.14	10.73	10.12	10.75	9.60	10.16	9.58	10.17	16.56	17.52	16.54	17.55	19.91	21.07	19.88	21.10
0.25	0.75	9.95	10.94	9.92	10.98	9.42	10.36	9.39	10.40	16.26	17.87	16.20	17.94	19.55	21.48	19.48	21.56
0	1	9.58	11.41	9.19	11.97	9.07	10.80	8.69	11.34	15.65	18.63	15.02	19.55	18.82	22.39	18.05	23.50

Note: the prices are calculated over a notional value of 100,000 Mus.

Table 8. Intuitionistic fuzzy prices of European put option prices on one-year zero coupon bonds.

		i=1				i=2				i=3				i=4			
α	β	$\underline{\Pi^{p}_{1_\alpha}}$	$\overline{\Pi^{p}_{1_\alpha}}$	$\underline{\Pi^{p'}_{1_\beta}}$	$\overline{\Pi^{p'}_{1_\beta}}$	$\underline{\Pi^{p}_{2_\alpha}}$	$\overline{\Pi^{p}_{2_\alpha}}$	$\underline{\Pi^{p'}_{2_\beta}}$	$\overline{\Pi^{p'}_{2_\beta}}$	$\underline{\Pi^{p}_{3_\alpha}}$	$\overline{\Pi^{p}_{3_\alpha}}$	$\underline{\Pi^{p'}_{3_\beta}}$	$\overline{\Pi^{p'}_{3_\beta}}$	$\underline{\Pi^{p}_{4_\alpha}}$	$\overline{\Pi^{p}_{4_\alpha}}$	$\underline{\Pi^{p'}_{4_\beta}}$	$\overline{\Pi^{p'}_{4_\beta}}$
1	0	10.43	10.43	10.43	10.43	9.87	9.87	9.87	9.87	7.44	7.44	7.44	7.44	8.53	8.53	8.53	8.53
0.75	0.25	10.29	10.57	10.28	10.58	9.74	10.00	9.73	10.01	7.34	7.54	7.34	7.55	8.41	8.64	8.41	8.65
0.5	0.5	10.14	10.73	10.12	10.75	9.60	10.16	9.58	10.17	7.24	7.66	7.23	7.67	8.29	8.77	8.28	8.79
0.25	0.75	9.95	10.94	9.92	10.98	9.42	10.36	9.39	10.40	7.11	7.81	7.08	7.84	8.14	8.94	8.11	8.98
0	1	9.58	11.41	9.19	11.97	9.06	10.80	8.69	11.34	6.84	8.14	6.56	8.54	7.83	9.33	7.51	9.79

Note: The prices are calculated over a notional value of 100,000 Mus.

5 | Conclusion

5.1 | General Considerations

Within the empirical applications of fuzzy subset theory, we can identify what is known as FROP. This fuzzy literature stream models the uncertainty and imprecision of parameters governing the random variation of asset prices to price options using FNs [8], [36]. In these applications, FNs capture the existence of epistemic uncertainty regarding the parameters governing the variation in underlying asset prices [9]. Most studies on the most common types of options analyzed in the FROP literature have examined the valuation of options on stocks and real options [8], [36].

This study extends the FROP in three aspects. First, uncertainty modelling is performed using IFNs. Second, regarding application settings, it focuses on fixed-income markets and assets with cash flows that are sensitive to interest rate fluctuations. This approach utilizes an analytical framework based on the BDT [40] yield curve. Third, we demonstrate the usefulness of a method that allows the estimation of short-term interest rate volatility through an IFN based on empirical data.

IFNs generalize FNs because they allow more information to be introduced into financial analysis. In our case, they allow us to capture not only epistemic uncertainty about the value of interest-rate volatility but also bipolar information about possible and unreliable values [10]. In this regard, it is noteworthy that our contribution is to the mainstream fuzzy literature that models uncertainty in financial economics using intuitionistic mathematics. It encompasses option pricing [35], [42–44] as well as other areas such as capital budgeting [60], [74], [75] and actuarial and risk analysis [76], [77].

The contributions of FROP to discrete-time option valuation are often located in the Cox et al. [6] binomial model CRR and options on stocks or real option. The option pricing literature has provided a great deal of binomial lattice frameworks to assess yield curves and interest rate derivatives [78]. However, to the best of

our knowledge, their extensions under the hypothesis of fuzzy uncertainty in the parameters governing interest rate movements have been summarized in [37–39]. Therefore, this study partially fills this research gap.

Furthermore, we use the methodology to estimate interest rate volatility through IFNs using empirical information [45]. The proposed approach leverages developments in transforming probabilistic information into possibilistic information [46], [51], [52] which justifies the adjustment of statistical parameters through possibility distributions overlaying confidence intervals with increasing significance levels [47], [55], [56]. In our approach, we adjust two possible distributions that serve to shape the intuitionistic estimation: one lower and one upper. The first allows the adjustment of the membership function and has a confidence interval support that can be obtained with the 95% rule. This encompasses reasonably extreme scenarios in terms of the values considered for volatility. The second possibility distribution, which allows obtaining the non-membership function, has a statistical confidence interval support that must be broader. For example, it can be adjusted using the 99.7% rule, which considers all possible scenarios regarding the value of the statistical parameter of interest.

While this study does not aim to demonstrate the empirical superiority of IFNs over classical fuzzy or probabilistic approaches, we argue that IFNs provide a richer representation of epistemic uncertainty. Unlike standard FNs, IFNs explicitly model both degrees of membership and non-membership, enabling a more nuanced depiction of decision-maker beliefs—especially in contexts where expert knowledge or incomplete information prevail. This added expressiveness may not always lead to improved pricing accuracy but does support more robust sensitivity and scenario analysis.

The proposed developments are useful for financial analysts, as the interpretation of the results, whether regarding volatility, prices, or interest rates, although they are IFNs, does not require knowledge of fuzzy mathematics. In fact, they rely largely on analytical tools traditionally used in business and finance, such as ex-post volatility [5] and/or the 95-99.7 rules of thumb [57]. Thus, the <1,0>-cut is a singleton that outperforms the quantity with greater trustworthiness, and is provided by the original BDT model. On the other hand, the <0,1>-cut indicates values associated with extreme scenarios. Thus, the 0-cut of $\mu_\sigma$shows soft severe projections, which can be assimilated to those associated with the 95% rule. On the other hand, the 1-cut of υ_σ indicates hard severe projections, which can be assimilated to the 99.7% rule. The use of intermediate $\langle \alpha, \beta \rangle$ −cuts allows the structuring of various scenarios according to their levels of membership and non-membership.

The results of this study are of particular interest to derivative instruments with cash flows sensitive to interest rates, such as bond options, caps, floors, and swaptions. These types of derivatives are traded in over-the-counter markets, where information about assets is more vague than in organized exchange markets. In fact, these instruments are often designed as "custom-made suits" for traders, so establishing fluctuation margins on prices that can reasonably be considered in negotiations between the buyer and seller of the instrument must be of special utility. Although we applied the intuitionistic estimate of the yield curve to the valuation of options on zero coupon bonds, it can be used to price any other type of derivative asset on interest rates with more complex structures.

5.2 | Limitations and Futher Research

This study is subject to several limitations that should be acknowledged. First, the proposed model relies on the assumption of normally distributed interest rate changes when constructing fuzzy confidence intervals. While this is a common assumption in the literature, it may not hold in all market conditions. Future research could address this by exploring distribution-free or nonparametric approaches.

Second, the estimation of volatility is based on historical data (ex-post volatility), which may not reflect forward-looking market expectations. Although this choice aligns with previous FROP studies and serves the illustrative purpose of this work, it limits the model's applicability for pricing in real-time financial markets. Integrating implied volatility extracted from market data would improve realism and empirical relevance.

Third, the analysis assumes crisp strike prices and fixed maturities. While this reflects standard contractual practice for bond options, it excludes the possibility of modeling uncertainty in contractual terms, which may be relevant in less regulated or incomplete markets.

Fourth, the proposed intuitionistic fuzzy BDT model increases computational complexity relative to its classical counterpart. Although tractability is maintained for short-to medium-term horizons, scalability to longer-term or high-frequency applications may require optimization strategies or approximations.

Finally, the model has not yet been tested against actual market data or benchmarked against traditional pricing methods. As a result, its empirical performance, particularly under volatile market conditions, remains to be validated.

These limitations opens several avenues for further research. One promising direction involves comparing the intuitionistic fuzzy framework with alternative approaches to modeling uncertainty, such as classical FNs, interval analysis, or probabilistic models. Such comparisons would allow for a deeper assessment of the added interpretive or predictive value of incorporating bipolar uncertainty into option pricing and interest rate modeling.

Additionally, the developments presented in this paper could inspire fuzzy or intuitionistic extensions of other interest rate models, such as those in [78], [79] by capturing the existence of uncertainty and imprecision in the parameters governing short-term rate fluctuations. Extending this methodology to multi-factor or continuous-time frameworks would further enhance its theoretical and practical relevance.

Another area for future research lies in the estimation of volatility itself. While the present study relies on historical (ex-post) volatility derived from observed interest rate fluctuations, a common approach in previous FROP studies [5], [14], [39], [48], future work could explore the use of implied volatility, extracted from market prices of derivatives. Within the FROP context, implied volatility has been estimated using fuzzy regression methods [80], [81] transformations of empirical distributions into possibility distributions [82–84] and GARCH-type models. These techniques could be adapted to generate intuitionistic fuzzy estimates of volatility, either through intuitionistic regression or by leveraging the interpretation of IFNs as bipolar possibility distributions.

Moreover, relaxing the distributional assumptions employed in constructing fuzzy confidence intervals, such as the assumption of normality, would enhance the robustness of the model. Future work could explore the use of nonparametric estimators, goodness-of-fit testing, or bootstrap-based methods to improve the empirical validity and generalizability of the results.

Although this study assumes crisp strike prices, fixed maturities and non credit risk, reflecting standard contractual conventions and trades on free default risk of fixed income intruments, incorporating uncertainty into these parameters may be relevant in markets where option terms are subject to renegotiation or expert judgment. From a computational standpoint, while the model remains tractable for short- to medium-term applications, future developments could investigate optimization techniques, parallelization, or closed-form approximations to improve its efficiency in large-scale or real-time scenarios.

Finally, empirical validation remains a key step. Comparing the model's output to observed market prices for bond options, as well as to classical BDT or alternative fuzzy models, would help gauge the practical utility of incorporating intuitionistic uncertainty into financial pricing.

Author Contribution

The paper has a sole author.

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Data Availability

The data used in this study are publicly available at https://www.ecb.europa.eu/home/html/index.en.html (European Central Bank).

Conflicts of Interest

The author declare no conflict of interest.

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