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SHORT NOTE

Rarity of the infinite chains in the tree of numerical semigroups

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Abstract
We prove that, for each fixed genus, the proportion of semigroups of that genus belonging to infinite chains in the semigroup tree approaches 0 as the genus grows to infinity. This means that most numerical semigroups have a finite number of descendants in the semigroup tree. This problem has been open since 2009.

Keywords Numerical semigroup · Semigroup tree · Infinite chains

1 Introduction

A numerical semigroup is a cofinite submonoid of the nonnegative integers $\mathbb{N}_0$. The nonnegative integers not belonging to a semigroup are called its gaps, and the number of gaps is its genus. In recent years there have been many efforts to count the number of numerical semigroups of each given genus, either theoretically [1, 2, 4, 10–13, 15, 17, 20–22] or computationally [3, 5, 7–9, 14]. One main tool is that of the tree of numerical semigroups. The root of this tree is the trivial numerical semigroup of genus 0, and at each depth of the tree there are the semigroups whose genus equals the depth. The parent of each non-trivial semigroup in the tree is the semigroup obtained by adding to the semigroup its largest gap, the so-called Frobenius number of the semigroup. This way, if one can explore the whole tree of semigroups, computing the number of semigroups of a given genus is equivalent to counting the number of nodes of the tree at the given depth.

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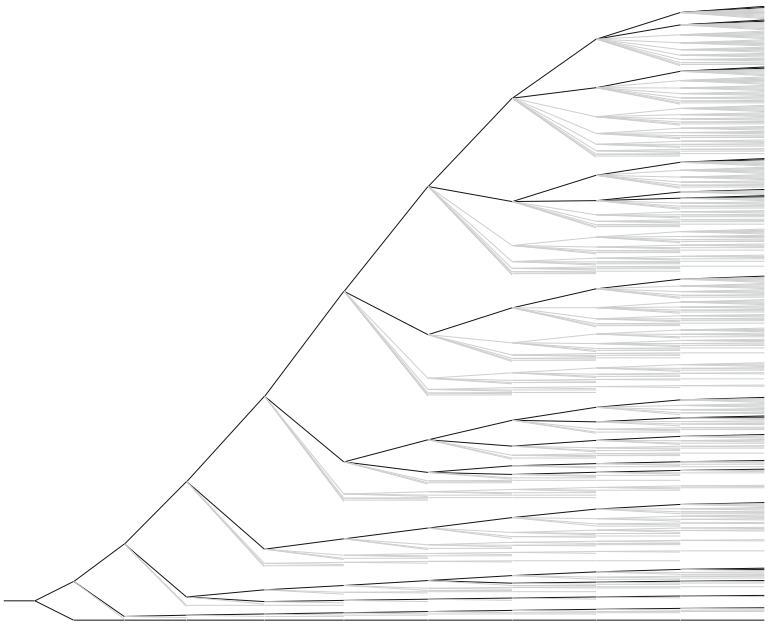


Fig. 1 Tree structure of the semigroup tree up to genus 11

An infinite chain in the semigroup tree is an infinite sequence of successively included numerical semigroups that are adjacent in the semigroup tree. They have been analyzed in [6, 18]. Clearly, a numerical semigroup has infinitely many descendants in the semigroup tree if and only if it lies in an infinite chain. Computation shows that semigroups in infinite chains are rare, although the number of nodes of the tree grows exponentially with the depth. In Fig. 1 one can see the tree structure of the semigroup tree up to genus 11. The edges corresponding to infinite chains have been highlighted. It can be appreciated that they constitute indeed, a very small proportion of the whole set of edges up to this genus. It is the object of this work to prove that, for each fixed genus g, the proportion of numerical semigroups of genus g in infinite chains among the whole set of numerical semigroups of genus g approaches 0 as the genus grows to infinity. This means that most numerical semigroups have a finite number of descendants in the semigroup tree.

The multiplicity of a numerical semigroup is its smallest nonzero element. Recent references analyze the expected multiplicity in terms of the genus of a semigroup as the genus grows to infinity, as well as the expected Frobenius number in terms of the multiplicity [16, 19]. Indeed, let $\gamma = (5+\sqrt{5})/10$. Fix $\varepsilon > 0$. It is stated in [16, Proposition 16] that the proportion of semigroups of genus g with multiplicity between $(\gamma-\varepsilon)g$ and $(\gamma+\varepsilon)g$ with respect to the whole set of numerical semigroups of genus g approaches 1 as the genus g grows to infinity. On the other hand, [16, Theorem 4(1)] shows that the proportion of semigroups of genus g with Frobenius number F satisfying $(2-\varepsilon)m < F < (2+\varepsilon)m$, where m is the multiplicity of the semigroup,

with respect to the whole set of numerical semigroups of genus g approaches 1 as the genus g grows to infinity. See also [19, Theorem 8].

The left elements of a numerical semigroup are all the elements of the semigroup that are smaller than the Frobenius number. It is proved in [6, 18] that a numerical semigroup belongs to an infinite chain if and only if its left elements are not coprime. In our proof we will use the aforementioned asymptotic results on the multiplicity and the Frobenius number of a numerical semigroup together with this characterization of semigroups in infinite chains.

2 Proof of the rarity of numerical semigroups in infinite chains

Let us denote the genus of a numerical semigroup S by $g(S)$, its multiplicity by $m(S)$, and its Frobenius number by $F(S)$. Given $a, b \in \mathbb{N}_0$ with $a < b$, we denote $[a, b) = \{x \in \mathbb{N}_0 \mid a \leq x < b\}$. Under this notation, the set of left elements of S is given by $L(S) = S \cap [0, F(S))$. Recall that $\gamma = (5 + \sqrt{5})/10$. Given a real number $\varepsilon > 0$ and a non-negative integer g, define the following:

$$\begin{aligned} S_g &= \{S : g(S) = g\} \\ n_g &= \#S_g \\ S_g^{\infty} &= \{S \in S_g : \gcd(L(S)) \neq 1\} \\ \mathcal{M}_g(\varepsilon) &= \{S \in S_g : \frac{m(S)}{g} \in (\gamma - \varepsilon, \gamma + \varepsilon)\} \\ \mathcal{F}_g(\varepsilon) &= \{S \in S_g : \frac{F(S)}{m(S)} \in (2 - \varepsilon, 2 + \varepsilon)\}. \end{aligned}$$

Lemma 1 ([16, Proposition 16]) *For any fixed* $\varepsilon > 0$, $\lim_{g\to\infty} \frac{\#\mathcal{M}_g(\varepsilon)}{n_g} = 1$.

Lemma 2 ([16, Theorem 4(1)]) *For any fixed* $\varepsilon > 0$, $\lim_{g\to\infty} \frac{\#\mathcal{F}_g(\varepsilon)}{n_g} = 1$.

Lemma 3 *For any fixed* $\varepsilon > 0$, $\lim_{g\to\infty} \frac{\#(\mathcal{M}_g(\varepsilon)\cap\mathcal{F}_g(\varepsilon))}{n_g} = 1$.

Proof The set S_g can be partitioned into disjoint subsets as follows.

$$S_g = (\mathcal{M}_g(\varepsilon) \cap \mathcal{F}_g(\varepsilon)) \sqcup (\mathcal{M}_g(\varepsilon) \backslash \mathcal{F}_g(\varepsilon)) \sqcup (\mathcal{F}_g(\varepsilon) \backslash \mathcal{M}_g(\varepsilon)) \sqcup (S_g \backslash (\mathcal{M}_g(\varepsilon) \cup \mathcal{F}_g(\varepsilon))).$$

Hence,

$$\#(\mathcal{M}_g(\varepsilon) \cap \mathcal{F}_g(\varepsilon)) = n_g - \#(\mathcal{M}_g(\varepsilon) \backslash \mathcal{F}_g(\varepsilon)) - \#(\mathcal{F}_g(\varepsilon) \backslash \mathcal{M}_g(\varepsilon)) - \#(S_g \backslash (\mathcal{M}_g(\varepsilon) \cup \mathcal{F}_g(\varepsilon))).$$

Now, by Lemmas 1 and 2,

$$\lim_{g\to\infty} \frac{\#(\mathcal{M}_g(\varepsilon) \backslash \mathcal{F}_g(\varepsilon))}{n_g} \leq \lim_{g\to\infty} \frac{\#(S_g \backslash \mathcal{F}_g(\varepsilon))}{n_g} = 0$$

$$\lim_{g\to\infty}\frac{\#(\mathcal{F}_g(\varepsilon)\backslash\mathcal{M}_g(\varepsilon))}{n_g}\leq\lim_{g\to\infty}\frac{\#(S_g\backslash\mathcal{M}_g(\varepsilon))}{n_g}=0$$

$$\lim_{g\to\infty}\frac{\#(S_g\backslash(\mathcal{M}_g(\varepsilon)\cup\mathcal{F}_g(\varepsilon)))}{n_g}\leq\lim_{g\to\infty}\frac{\#(S_g\backslash\mathcal{M}_g(\varepsilon))}{n_g}=0.$$

It follows that

$$\lim_{g\to\infty}\frac{\#(\mathcal{M}_g(\varepsilon)\cap\mathcal{F}_g(\varepsilon))}{n_g}=1.$$

□

Lemma 4 ([6, Theorem 10],[18, Lemma 3.4]) *A numerical semigroup S belongs to an infinite chain if and only if the elements in $L(S)$ are not coprime.*

Hence, S_g^∞ is, indeed, the number of numerical semigroups of genus g belonging to infinite chains. Now, the next theorem proves that, as the genus tends to infinity, almost all semigroups are not contained in any infinite chain.

Theorem 5 *With the same notations as before,* $\lim_{g\to\infty}\frac{\#S_g^\infty}{n_g}=0$.

Proof Fix $\varepsilon<0.01$.

Let us first prove that $\mathcal{M}_g(\varepsilon)\cap\mathcal{F}_g(\varepsilon)\subseteq S_g\backslash S_g^\infty$. Indeed, take a numerical semigroup $S\in\mathcal{M}_g(\varepsilon)\cap\mathcal{F}_g(\varepsilon)$. Let m and F denote the multiplicity and Frobenius number of S, and let $[m,F)$ denote the integer interval between m and $F-1$. The number of gaps of S in $[m,F)$ is equal to $g-m$, which is less than $(1-\gamma+\varepsilon)g$. The length of $[m,F)$ is $F-m$, which exceeds $(1-\varepsilon)m>(1-\varepsilon)(\gamma-\varepsilon)g$. Thus, the proportion of gaps in $[m,F)$ is upper bounded by

$$\frac{(1-\gamma+\varepsilon)g}{(1-\varepsilon)(\gamma-\varepsilon)g}=\frac{(1-\gamma+\varepsilon)}{(1-\varepsilon)(\gamma-\varepsilon)}<\frac{1-0.72+0.01}{(1-0.01)(0.72-0.01)}<0.42.$$

If $g\geq 2$, then $[m,F)$ has length at least $\lceil(1-\varepsilon)(\gamma-\varepsilon)2\rceil\geq\lceil(1-0.01)(0.72-0.01)2\rceil=2$. Therefore, there must exist at least two consecutive integers in $S\cap[m,F)$. So, $\gcd(L(S))=1$.

Now, by the previous inclusion and Lemma 3,

$$\lim_{g\to\infty}\frac{\#S_g^\infty}{n_g}=1-\lim_{g\to\infty}\frac{\#(S_g\backslash S_g^\infty)}{n_g}\leq 1-\lim_{g\to\infty}\frac{\#(\mathcal{M}_g(\varepsilon)\cap\mathcal{F}_g(\varepsilon))}{n_g}=0.$$

□

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Data Availability The graph has been drawn using the drawsgtree tool, which can be downloaded from https://github.com/mbrasamoros/drawsgtree.

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